

Complete solution of the Diophantine Equation $x^2 + 5^a \cdot 11^b = y^n$

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Abstract

The title equation is completely solved in integers (n, x, y, a, b) , where $n \geq 3$, $\gcd(x, y) = 1$ and $a, b \geq 0$. The most difficult stage of the resolution is the explicit resolution of a quintic Thue-Mahler equation. Since it is for the first time -to the best of our knowledge- that such an equation is solved in the literature, we make a detailed presentation of the resolution; this gives our paper also an expository character.

1 Introduction

The title equation belongs to the general class of Diophantine equations of the form

$$x^2 + D = y^n, \quad x, y \geq 1, \quad n \geq 3, \quad (1.1)$$

where D is a positive integer all whose prime factors belong to a finite set S of at least two distinct primes. All solutions of the Diophantine equation (1.1) have been determined for various sets S : In [21] for $S = \{2, 3\}$, in [22] for $S = \{2, 5\}$, in [10] for $S = \{2, 11\}$, in [23] for $S = \{2, 13\}$, in [13] for $S = \{2, 17\}$, $S = \{2, 29\}$, $S = \{2, 41\}$, in [32] for $S = \{2, 19\}$. Note that, in all these cases, $S = \{2, p\}$, where p is an odd prime. The case of $S = \{2, p\}$, with a general odd prime p , was recently studied by H. Zhu, M. Le, G. Soydan and A. Togbé [38], who gave all the solutions of $x^2 + 2^a p^b = y^n$, $x \geq 1, y > 1$, $\gcd(x, y) = 1$, $a \geq 0, b > 0$, $n \geq 3$ under some conditions.

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Several papers deal with the Diophantine equation (1.1) when S contains at least two distinct odd primes. Thus, all solutions of the Diophantine equation (1.1) were given in [2] for $S = \{5, 13\}$, in [28] for $S = \{5, 17\}$, in [30], [31] for $S = \{7, 11\}$ -except for the case when ax is odd and b is even-, in [6] for $S = \{11, 17\}$, in [17] for $S = \{2, 5, 13\}$, in [9] for $S = \{2, 3, 11\}$, in [15] for $S = \{2, 5, 17\}$. In [27], Pink gave all the non-exceptional solutions of the equation (1.1) (according to the terminology of that paper) for $S = \{2, 3, 5, 7\}$. A survey of these and many others can be found in [6], [1]. Very recently, the equations with $S = \{2, 3, 17\}$ and $S = \{2, 13, 17\}$ were solved in [16].

In [11], I.N. Cangul, M. Demirci, G. Soydan and N. Tzanakis gave the complete solution (n, a, b, x, y) of the Diophantine equation (1.1) for $S = \{5, 11\}$ when $\gcd(x, y) = 1$, except for the case when abx is odd. In this paper we treat this remaining case, proving thus the following:

Theorem 1.1. *For the integer solutions of the equation*

$$x^2 + 5^a 11^b = y^n, \quad n \geq 3, x, y \geq 1, \gcd(x, y) = 1, a, b \geq 0, \quad (1.2)$$

the following hold:

If $n = 3$, the only solutions are: $(a, b, x, y) = (0, 1, 4, 3), (0, 1, 58, 15), (0, 2, 2, 5), (0, 3, 9324, 443), (1, 1, 3, 4), (1, 1, 419, 56), (2, 3, 968, 99), (3, 1, 37, 14), (5, 5, 36599, 1226)$.

If $n = 4$ there are no solutions (a, b, x, y) and, for $n = 6$, the only integer solution is $(a, b, x, y) = (1, 1, 3, 2)$.

If $n = 5$ or $n \geq 7$, the equation has no integer solutions (a, b, x, y) .

The proof of Theorem 1.1 is already accomplished in [11, Theorem 1] for the following cases: (A) $n = 3, 4, 6$, and (B) $n \geq 5, n \neq 6$ and either (i) ab is odd and x is even, or (ii) at least one of a, b is even. Therefore, what remains is:

To prove that, if $n \geq 5$ is prime, then the equation (1.2) has no solution (a, b, x, y) with abx odd.

This paper has two objectives, the first one being displayed above. Second is the systematic discussion in Section 3 of the resolution of the quintic Thue-Mahler equation (3.5) which, along with the three Appendices (see “Plan of the paper” below) lends also an expository character to the paper, as it presents in detail the application of the method of N. Tzanakis & B.M.M. de Weger [34] to the explicit resolution of a *quintic* Thue-Mahler equation. To the best of our knowledge, in the literature it is the first example of explicit resolution of a *quintic* Thue-Mahler equation. Indeed, in [34], the worked example is a *cubic* Thue-Mahler equation; those days –almost 25 years ago– the

available software was not as developed as to support the application of the method to a quintic Thue-Mahler equation; even until today, only very few works are published in which Thue-Mahler equations are *explicitly* solved and none of them deals with a quintic equation; more specifically: In [11], I.N. Cangul, M. Demirci, G. Soydan and N. Tzanakis need to solve –successfully– a quartic Thue-Mahler equation. In [19], Dohyeong Kim proposes a method different from that of [34] – with many examples– for the explicit resolution of *cubic* Thue-Mahler equations, which exploits the modularity of elliptic curves over \mathbb{Q} . M.A. Bennett and S.R. Dahmen [4], in their study of *generalized superelliptic* equations need to consider some special classes of Thue-Mahler equations. These are closely related to the so-called *Klein forms*, which are defined as binary forms of the following shape: $F(x, y) := F_n(ax + by, cx + dy)$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\overline{\mathbb{Q}})$, $n \in \{2, 3, 4, 5\}$ and $F_2(x, y) = xy(x + y)$, $F_3(x, y) = y(x^3 + y^3)$, $F_4(x, y) = xy(x^4 + y^4)$, $F_5(x, y) = xy(x^{10} - 11x^5y^5 - y^{10})$. Thue-Mahler equations whose left-hand side is a Klein form are considered. In the case of *cubic* Klein forms, Bennett and Dahmen, implemented the method of [34]; for the purposes of their paper, explicit resolution of higher degree Thue-Mahler equations was not necessary.

Finally, we mention Kyle Hambrook’s M.Sc. thesis [18], where the method of [34] is revisited, certain improvements are included and, most importantly, a long¹ MAGMA program is developed for the automatic resolution of the general Thue-Mahler equation; the program needs as its input only the coefficients and the primes of the equation. No examples of Thue-Mahler equations of degree greater than three are discussed. As we checked, the program runs very successfully with “reasonable” cubic Thue-Mahler equations. This work is, certainly, a good contribution to the project of the automatic resolution of Thue-Mahler equations. However, in the case of our quintic Thue-Mahler equation (3.5), it took 72 days on an Apple computer with the following characteristics: Processor Intel i5, 2.5 GHz, 4GB RAM, 1600 MHz DDR3. Therefore, we preferred to develop also our own MAGMA program, far less automatic than that of Hambrook, which needs human intervention at various points. With this program, the resolution of the quintic Thue-Mahler equation took less than 2 h 20’. Besides this huge difference in computation time, this “primitive” type of computer-aided resolution, however, has the advantage that it allows a rather transparent presentation of the very complicated resolution. It is our belief that the experience in the “technical details” needed for the development of a very satisfactory “Thue-Mahler automatic solver” requires the resolution of quite a number of specific Thue-Mahler equations (over \mathbb{Z}) of degree ≤ 6 (at least), with corresponding number fields of various types.²

¹More than 200 pages!

²We mention here that very satisfactory automatic Thue solvers are included, for example, in PARI [26] and MAGMA [3] since long time, and are based on Bilu-Hanrot’s improvement [7] of Tzanakis-de Weger [33] method for solving Thue equations. The development of an automatic Thue solver is, certainly, a difficult job but, anyway, much easier than an analogous job for a Thue-Mahler equation. We use the MAGMA Thue solver in Subsection 3.2.9.

Plan of the paper. In Section 2 we prove that equation (1.2) has no solutions with abx odd and prime $n \geq 7$. Thus, we are reduced to proving that our equation has no solutions with abx odd and $n = 5$. This is accomplished in Section 3 which is the heart of the paper and is divided into two subsections. In Subsection 3.1, using standard algebraic Number Theory, we reduce the equation $x^2 + 5^a 11^b = y^5$ -with $\gcd(x, y) = 1$ and abx odd- to the quintic Thue-Mahler equation (3.5), whose right-hand side is $-2^5 3^4 5^{z_1} 11^{z_2}$, where $z_1 = (a - 1)/2$ and $z_2 = (b - 1)/2$ are now our non-negative unknown integers. Then, Subsection 3.2 is devoted to the resolution of that Thue-Mahler equation, quite a complicated task. In order to make the exposition of our resolution as clear as possible, we divided Subsection 3.2 into nine (sub)subsections from (sub)Subsection 3.2.1 through (sub)Subsection 3.2.9.

- In Subsection 3.2.1, using standard arguments from algebraic Number Theory along with the valuable routines of MAGMA [3], we reduce our quintic Thue-Mahler equation to the ideal equations (3.7) and (3.8), in the right-hand side of which appear the unknown non-negative integers z_1 and z_2 .
- In Subsection 3.2.2, working 5-adically, we prove that $z_1 \leq 27$ (implying $a \leq 55$).
- In Subsection 3.2.3 we work 11-adically. Making also use of the upper bound $z_1 \leq 27$, we are led to the following situation: Instead of solving one (quintic) Thue-Mahler equation in which the exponents of the primes 5 and 11 are among the unknowns, we are led to solving 28 similar Thue-Mahler equations, with all having the same left-hand side and right-hand sides in which only the exponent of the prime 11 is among the unknowns. This is certainly a profit; these 28 equations can be treated as one equation, namely, equation (3.16). In this last equation, besides the unknown integer $z_2 = (b - 1)/2$, three more unknown integers a_1, \dots, a_4 make their appearance; these are the exponents of the four fundamental units of the quintic field related to the Thue-Mahler equation.
- In Subsection 3.2.4 the aforementioned equation (3.16) leads to the three-term S -unit equation (3.17); this is the basic step towards the use of *Linear Forms in Logarithms* in both the real/complex and the p -adic sense; consequently:
- In Subsection 3.2.5 we are in a position to apply a powerful result of Kunrui Yu (Theorem 3.2 in this paper) which, given the algebraic numbers $\alpha_1, \dots, \alpha_n$ and a prime p , provides an upper bound for the p -adic valuation of $\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$, for any $b_1, \dots, b_n \in \mathbb{Z}$, in terms of $\log \max\{3, |b_1|, \dots, |b_n|\}$.³ Combining the result of this application with the instructions in p. 238 of [34] we manage to bound z_2 in terms of $\log \max\{z_2, |a_1|, \dots, |a_4|\}$. We remark here that, in order to conform with the notation of [34], the recipes of which we follow very closely, we denote z_2 by n_1 .
- In Subsection 3.2.6 we apply another strong result due to E.M. Matveev (Theorem 3.3 in this paper)

³This is, actually, equivalent to giving a lower bound of the p -adic absolute value for a linear form in the p -adic logarithms of $\alpha_1, \dots, \alpha_n$.

which, given the algebraic numbers $\alpha_1, \dots, \alpha_n$, provides⁴ a lower bound for $|b_1 \log \alpha_1 + \dots + b_n \log \alpha_n|$, for any $b_1, \dots, b_n \in \mathbb{Z}$, in terms of $\log \max\{3, |b_1|, \dots, |b_n|\}$. Applying this theorem in our case and combining the result with the detailed instructions of [34, §§ 10,11], we obtain a numerical upper bound for $H = \max\{n_1, |a_1|, |a_2|, |a_3|, |a_4|\}$; see (3.23). Then, this upper bound of H , in combination with the result of the previous Subsection 3.2.5, gives a specific numerical upper bound for n_1 , which is considerably smaller than H ; see (3.24). However, both upper bounds are huge, of the size of 10^{43} and 10^{32} , respectively and need to be reduced to a manageable size, as discussed in [34, Section 13].

- In Subsection 3.2.7 we apply the so-called “ p -adic reduction step”, which is described in detail in Sections 12,14 and 15 of [34] and reduce the upper bound for n_1 to $n_1 \leq 207$.
- In Subsection 3.2.8 we combine this extremely smaller upper bound with the (still remaining) huge upper bound of H , and do the “real reduction step”, following the instructions of [34, Section 16]. With this step we get the bound $H \leq 231$.
- In the final Subsection 3.2.9 we discuss how we proceed with a further reduction, by successively repeating the “ p -adic reduction” and the “real reduction” steps two more times, until we obtain the bound $(z_2 =) n_1 \leq 21$ and $H \leq 34$. At this point we don’t need the bound $H \leq 34$; as explained in Subsection 3.2.9, we are left with the task of solving 560 Thue equations (3.5), whose right-hand sides runs through the set $\{-2^5 3^4 5^{z_1} 11^{z_2} : 0 \leq z_1 \leq 27, z_2 = 0 \text{ or } 3 \leq z_2 \leq 21\}$; note that the left-hand sides of all these Thue equations are identical. For their solution we use MAGMA’s implementation of Bilu & Hanrot’s method [7]. It turns out that no solutions exist and this completes the proof that equation (1.2) with abx odd and $n = 5$ has no solutions. Since in Section 2 we have also proved that equation (1.2) with abx odd and $n > 5$ has no solutions, we have completed the proof of Theorem 1.1.

In the Appendices A through B at the end of the paper we collect some theoretical facts and give some information about how these are realized in practice with the use of MAGMA [3]. We also give the results of a few computations. The huge algebraic numbers in Appendix B are not strictly necessary; however, they are useful in giving the reader a sense of what “monsters” are involved in such a task. We hope that the appendices will make transparent our way of work and friendly the reading of our paper.

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⁴Under one or two mild conditions.

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2 Equation (1.2) with abx odd and prime $n \geq 7$

Proposition 2.1. *Equation (1.2) has no solutions with xab odd and prime $n \geq 7$.*

Proof. We assume that a solution (x, a, b, n, y) in which xab is odd and n is a prime ≥ 7 does exist, and we put $a = na_1 + \alpha$ and $b = nb_1 + \beta$, where $0 \leq \alpha, \beta < n$, so that our equation becomes

$$5^\alpha 11^\beta (-5^{a_1} 11^{b_1})^n + y^n = x^2. \quad (2.1)$$

Without loss of generality we assume that $x \equiv 1 \pmod{4}$. According to the notation etc of [29, Section 14], this is a ternary equation of signature $(n, n, 2)$, so that it falls under the scope of the recipe described in [29, §14.2]. Accordingly, we have the following table which shows how we will apply that recipe in our case.

Table 1: Application of the recipe in §14.2 of [29]

Notations/conditions in [29, §14.2]	Interpretations in this paper
A	$5^\alpha 11^\beta$
B	1
C	1
x	$-5^{a_1} 11^{b_1}$
y	y
z	x
general prime q	general prime q
p	n
$\text{ord}_q(B) < p$	trivially satisfied
$\text{ord}_q(A) < p$	$\alpha, \beta < p$
C square-free	trivially satisfied

Since y is even, $x \equiv 1 \pmod{4}$ and $n \geq 7$, our equation falls in case (v) of [29, §14.2] and we deal with the elliptic curve

$$E_3 : Y^2 + XY = X^3 + \frac{x-1}{4}X^2 + \frac{y^n}{64}X.$$

According to a result of Bennett and Skinner [5, Lemma 2.1] (or [29, Theorem 16]), the discriminant

and conductor of this elliptic curve are, respectively

$$\Delta_3 = -2^{-12} 5^\alpha 11^\beta (5^{a_1} 11^{b_1} y^2)^n = -2^{-12} 5^a 11^b y^{2n}, \quad N_3 = 55 \operatorname{Rad}(y) = 5 \cdot 11 \prod_{q|y} q,$$

where in the last product q is prime. According to [29, Theorem 16 (c)], there exists a newform f of level $N_n = 2 \cdot 5 \cdot 11 = 110$, such that $E_3 \sim_n f$ (E_3 arises from $f \bmod n$; see [29, §5]).

A computation using MAGMA returns three rational newforms of level 110, namely⁵

$$\begin{aligned} f_1 &= q - q^2 + q^3 + q^4 - q^5 - q^6 + 5q^7 - q^8 - 2q^9 + q^{10} + q^{11} + O(q^{12}) \\ f_2 &= q + q^2 + q^3 + q^4 - q^5 + q^6 - q^7 + q^8 - 2q^9 - q^{10} - q^{11} + O(q^{12}) \\ f_3 &= q + q^2 - q^3 + q^4 + q^5 - q^6 + 3q^7 + q^8 - 2q^9 + q^{10} + q^{11} + O(q^{12}) \end{aligned}$$

and a non-rational newform

$$f_4 = q - q^2 + \alpha q^3 + q^4 + q^5 - \alpha q^6 - \alpha q^7 - q^8 + (5 - \alpha)q^9 - q^{10} - q^{11} + O(q^{12}),$$

where $\alpha^2 + \alpha - 8 = 0$, along with its conjugate newform.

Now we apply [29, Proposition 9.1] to $E = E_3$ and $f = f_i$, $i = 1, 2, 3, 4$. Our notation refers to that Proposition. Since $(X, Y) = (0, 0)$ is a 2-torsion point on E_3 , we take $t = 2$. Also $N' = 110$ and we choose the prime $\ell = 3$, noting that $\ell \nmid N'$ and $\ell^2 \nmid N_3$. Then,

$$S_3 = \{a \in \mathbb{Z} : -2\sqrt{3} \leq a \leq 2\sqrt{3}, a \text{ is even}\} = \{-2, 0, 2\}.$$

Also, denoting by c_{3i} the coefficient of q^3 in the newform f_i , we compute

$$B_3(f_i) = (4^2 - c_{3i}^2) \prod_{a \in S_3} (a - c_{3i}) = \begin{cases} 3^2 \cdot 5 & \text{if } i = 1, 2 \\ -3^2 \cdot 5 & \text{if } i = 3. \end{cases}$$

For the newform f_4 we compute

$$B_3(f_4) = 3 \cdot \operatorname{Norm}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(4^2 - \alpha^2) \prod_{k=-1}^1 \operatorname{Norm}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(2k - \alpha) = -3^3 \cdot 2^9.$$

According to the conclusion of [29, Proposition 9.1], n must divide $B_3(f_i)$ for some $i \in \{1, 2, 3, 4\}$, which is impossible since we assumed that n is a prime ≥ 7 . \square

⁵Below q denotes the “ q -variable” of the modular form and has nothing to do with primes.

3 Equation (1.2) with abx odd and $n = 5$

3.1 Reduction to the Thue-Mahler equation (3.5)

In view of Proposition 2.1, we are left with $n = 3, 5$. The case $n = 3$ is already solved completely; see [11, Proposition 2]. In particular, the only solutions with abx odd are the following:

$$(a, b, x, y) = (1, 1, 3, 4), (1, 1, 419, 56), (3, 1, 37, 14), (5, 5, 36599, 1226).$$

It remains to treat the title equation when $n = 5$ and xab is odd. We write our equation

$$x^2 + 55z^2 = 2^5 y_1^5, \quad x \equiv 1 \pmod{4}, \quad y = 2y_1, \quad z = 5^{(a-1)/2} 11^{(b-1)/2} \quad (3.1)$$

and work in the field

$$L = \mathbb{Q}(\rho), \quad \rho = \frac{1 + \sqrt{-55}}{2} \quad (\rho^2 - \rho + 14 = 0).$$

Using either PARI-GP [26] or MAGMA [3] we can obtain the following facts about the number-field L :

- The class-number is 4.
- $\langle 2 \rangle = \mathfrak{p}_2 \mathfrak{p}'_2$, where $\mathfrak{p}_2 = \langle 2, \rho \rangle$, $\mathfrak{p}'_2 = \langle 2, 3 + \rho \rangle$.
- The order of the ideal-class of both \mathfrak{p}_2 and \mathfrak{p}'_2 in the ideal-class group is 4. More specifically, $\mathfrak{p}_2^4 = \langle 2 - \rho \rangle$ and $\mathfrak{p}'_2 = \langle 1 + \rho \rangle$.
- $\langle \rho \rangle = \mathfrak{p}_2 \langle 7, \rho \rangle$.

From (3.1) we obtain the ideal equation

$$\mathfrak{p}_2^5 \mathfrak{p}'_2{}^5 \langle y_1 \rangle^5 = \langle x + z\sqrt{-55} \rangle \langle x - z\sqrt{-55} \rangle = \langle x - z + 2z\rho \rangle \langle x + z - 2z\rho \rangle \quad (3.2)$$

Our first observation is that no prime ideal factor over an odd rational prime can divide both ideal factors in the right-hand side of (3.2). Indeed, if \mathfrak{p} is a prime ideal over an odd rational prime, then $\mathfrak{p} | 2x$, hence $\mathfrak{p} | x$. But $\mathfrak{p} | \langle x + z\sqrt{-55} \rangle$, hence $\mathfrak{p} | z\sqrt{-55}$. It follows that $\mathfrak{p} | 55$, which contradicts $\gcd(x, 55) = 1$.

Next we observe that 2 divides both $x - z + 2z\rho$ and $x + z - 2z\rho$. From $\langle \rho \rangle = \mathfrak{p}_2 \langle 7, \rho \rangle$ we see that $\langle 2z\rho \rangle = \mathfrak{p}_2^2 \mathfrak{p}'_2 \times (\text{ideal relatively prime to } 2)$. Also $x + z \equiv 2 \pmod{4}$ shows that $\text{ord}_{\mathfrak{p}_2}(x + z) = 1 = \text{ord}_{\mathfrak{p}'_2}(x + z)$. As a consequence, $\text{ord}_{\mathfrak{p}_2}(x + z - 2z\rho) = 1 = \text{ord}_{\mathfrak{p}'_2}(x + z - 2z\rho)$, which implies that $\mathfrak{p}_2 \parallel \langle x + z - 2z\rho \rangle$ and $\mathfrak{p}'_2 \parallel \langle x - z + 2z\rho \rangle$. Similarly, starting from $x - z \equiv 0 \pmod{4}$, we conclude that $\mathfrak{p}'_2 \parallel \langle x - z - 2z\rho \rangle$.

Combining the above small observations with (3.2), we conclude that

$$\langle x - z + 2z\rho \rangle = \mathfrak{p}'_2 \mathfrak{p}_2^4 \mathfrak{a}_1^5, \quad \langle x + z - 2z\rho \rangle = \mathfrak{p}_2 \mathfrak{p}'_2{}^4 \mathfrak{a}_2^5, \quad (3.3)$$

where $\mathfrak{a}_1, \mathfrak{a}_2$ are relatively prime (integral) ideals, such that $\mathfrak{a}_1 \mathfrak{a}_2 = \langle y_1 \rangle$.

The first equation (3.3) becomes

$$\langle x - z + 2z\rho \rangle = \mathfrak{p}'_2 \mathfrak{a}_1^5 \langle 2 - \rho \rangle.$$

Since the class-number is 4, the above ideal equation implies the ideal-class equation $[1] = [\mathfrak{p}'_2][\mathfrak{a}_1] = [\mathfrak{p}'_2 \mathfrak{a}_1]$. Therefore, $\mathfrak{p}'_2 \mathfrak{a}_1$ is a principal ideal, so that, on multiplying both sides of the above displayed equation by \mathfrak{p}_2^4 and setting $\mathfrak{p}'_2 \mathfrak{a}_1 = \langle u + v\rho \rangle$ ($u, v \in \mathbb{Z}$) we finally arrive at the following (element) equation

$$(1 + \rho)(x - z + 2z\rho) = (2 - \rho)(u + v\rho)^n. \quad (3.4)$$

In (3.4) we equate coefficients of ρ in both sides, as well as rational parts, obtaining thus the following two relations:

$$3u^5 + 65vu^4 - 290v^2u^3 - 2110v^3u^2 + 975v^4u + 3149v^5 = -32 \cdot 5^{(a-1)/2} 11^{(b-1)/2},$$

$$23u^5 - 355vu^4 - 3930v^2u^3 + 6010v^3u^2 + 30515v^4u - 2311v^5 = -32x.$$

By multiplying both sides of the first displayed equation by 3^4 we get

$$\begin{aligned} (3u)^5 + 65v(3u)^4 - 870v^2(3u)^3 - 18990v^3(3u)^2 + 26325v^4(3u) + 255069v^5 \\ = -2^5 \cdot 3^4 \cdot 5^{(a-1)/2} 11^{(b-1)/2} \end{aligned}$$

In order to conform precisely with the notations of [34] the method of which we will apply in this section, we set

$$(3u, v) = (x, y), \quad z_1 = (a - 1)/2, \quad z_2 = (b - 1)/2,$$

so that

$$\text{Norm}_{F/\mathbb{Q}}(x - y\theta) = -2^5 \cdot 3^4 \cdot 5^{z_1} 11^{z_2}, \quad (3.5)$$

where $F = \mathbb{Q}(\theta)$ with $g(\theta) = 0$ and

$$g(t) = t^5 + 65t^4 - 870t^3 - 18990t^2 + 26325t + 255069, \quad (3.6)$$

with (polynomial) discriminant $D_\theta = 2^{32} 3^{12} 5^{11} 11^6$.

3.2 Resolution of the Thue-Mahler equation (3.5)

3.2.1 From equations (3.5) to ideal equations (3.7) and (3.8)

We need the following arithmetical data for the number field F .

- F is a totally real field with class-number 1.
- An integral basis is $1, \beta_2, \beta_3, \beta_4, \beta_5$, where

$$\beta_2 = \frac{1}{2}(\theta + 1), \quad \beta_3 = \frac{1}{24}(\theta^2 + 2\theta + 9),$$

$$\beta_4 = \frac{1}{7920}(\theta^3 + 227\theta^2 + 3603\theta + 3969), \quad \beta_5 = \frac{1}{95040}(\theta^4 + 8\theta^3 + 1410\theta^2 + 46512\theta + 9909).$$

- A quadruple of fundamental units is the following:

$$\epsilon_1 = \frac{1}{15840}(\theta^4 + 62\theta^3 - 852\theta^2 - 1806\theta + 6435) \quad (\text{Norm}(\epsilon_1) = 1)$$

$$\epsilon_2 = \frac{1}{95040}(-\theta^4 - 104\theta^3 + 558\theta^2 + 35280\theta + 108027) \quad (\text{Norm}(\epsilon_2) = -1)$$

$$\epsilon_3 = \frac{1}{23760}(\theta^4 + 77\theta^3 + 243\theta^2 + 99\theta - 10260) \quad (\text{Norm}(\epsilon_3) = 1)$$

$$\epsilon_4 = \frac{1}{95040}(7\theta^4 + 596\theta^3 + 5730\theta^2 - 25596\theta - 210897) \quad (\text{Norm}(\epsilon_4) = 1)$$

- Prime factorization of 2:

$$2 = -\epsilon_1^2 \epsilon_2^2 \epsilon_3^2 \epsilon_4^2 \pi_2^5,$$

$$\pi_2 = \frac{1}{95040}(17\theta^4 + 1048\theta^3 - 18486\theta^2 - 271440\theta + 1590381) \quad (\text{Norm}(\pi_2) = -2)$$

- Prime factorization of 3:

$$3 = -\pi_{31}\pi_{32}$$

$$\pi_{31} = \frac{1}{95040}(13\theta^4 + 896\theta^3 - 7806\theta^2 - 280008\theta - 861975) \quad (\text{Norm}(\pi_{31}) = 3)$$

$$\pi_{32} = \frac{1}{31680}(\theta^4 + 68\theta^3 - 810\theta^2 - 22428\theta + 105489) \quad (\text{Norm}(\pi_{32}) = -3^4)$$

- Prime factorization of 5:

$$5 = -\epsilon_1 \epsilon_2^{-1} \epsilon_3^2 \epsilon_4^2 \pi_5^5$$

$$\pi_5 = \frac{1}{95040}(13\theta^4 + 896\theta^3 - 7806\theta^2 - 280008\theta - 671895) \quad (\text{Norm}(\pi_5) = 5)$$

- Prime factorization of 11:

$$\begin{aligned}
11 &= \epsilon_1^{-1} \epsilon_2^{-1} \epsilon_4^{-1} \pi_{111}^2 \pi_{112}^2 \pi_{113} \\
\pi_{111} &= \frac{1}{47520} (-\theta^4 - 56\theta^3 + 1554\theta^2 + 22104\theta + 43119) \quad (\text{Norm}(\pi_{111}) = 11) \\
\pi_{112} &= \frac{1}{95040} (\theta^4 + 68\theta^3 - 810\theta^2 - 6588\theta + 57969) \quad (\text{Norm}(\pi_{112}) = 11) \\
\pi_{113} &= \frac{1}{31680} (-13\theta^4 - 896\theta^3 + 7806\theta^2 + 280008\theta + 798615) \quad (\text{Norm}(\pi_{113}) = -11)
\end{aligned}$$

The above information combined with (3.5) easily implies that we have the following possibilities:

$$\langle x - y\theta \rangle = \langle \pi_2 \rangle^5 \langle \pi_{31} \rangle^4 \langle \pi_5 \rangle^{z_1} \langle \pi_{111} \rangle^{w_1} \langle \pi_{112} \rangle^{w_2} \langle \pi_{113} \rangle^{w_3} \quad (3.7)$$

$$\langle x - y\theta \rangle = \langle \pi_2 \rangle^5 \langle \pi_{32} \rangle \langle \pi_5 \rangle^{z_1} \langle \pi_{111} \rangle^{w_1} \langle \pi_{112} \rangle^{w_2} \langle \pi_{113} \rangle^{w_3} \quad (3.8)$$

where, in both cases,

$$w_1 + w_2 + w_3 = z_2. \quad (3.9)$$

3.2.2 Treating 5-adically equations (3.7) and (3.8)

From the ideal equations (3.7) and (3.8) we will compute an upper bound for the unknown exponent z_1 , using the “Second Corollary of Lemma 1” in Section 5 of [34]. As a consequence, the equation (3.5) will be replaced by a rather small number of similar equations in which only the exponent z_2 will be unknown; this is certainly a gain. With the notations of Sections 3,5 of [34] we have in our case:

Table 2: Application of “Second Corollary of Lemma 1” in [34] when $p = 5$

Notations in [34, §§ 3,5]	Interpretations in this case and [references in this paper]
p	5
$g(t)$ (§3)	$t^5 + 65t^4 - 870t^3 - 18990t^2 + 26325t + 255069$ [(3.6)]
m (§3)	1; $g(t)$ is irreducible over \mathbb{Q}_5 , hence $g(t) = g_1(t)$
\mathfrak{p}_1, e_1, d_1 (§3)	$\langle \pi_5 \rangle, 5, 1$ [page 10]
e (§5)	5
<i>continued on next page</i>	

<i>continued from previous page</i>	
Notations in [34, §§ 3,5]	Interpretations in this paper
D_θ (§5)	$2^{32}3^{12}5^{11}11^6$ [(3.6)]

In view of the fact that $m = 1$ and $e_1 = 5$, the “Second Corollary of Lemma 1” in [34] implies that $z_1 = \text{ord}_{\pi_5}(x - y\theta) \leq \frac{1}{2}e \cdot \text{ord}_5(D_\theta) = 55/2$, hence,

$$z_1 \leq 27, \quad \text{implying } a \leq 55. \quad (3.10)$$

3.2.3 Treating 11-adically equations (3.7) and (3.8)

Once again we use the notations of Sections 3,5 of [34]. Now $g(t) = g_1(t)g_2(t)g_3(t)$ is the factorization of $g(t)$ into irreducible polynomials of $\mathbb{Q}_{11}[t]$, where

$$g_1(t) = t^2 + (3 + 3 \cdot 11 + 8 \cdot 11^2 + 9 \cdot 11^3 + 5 \cdot 11^4 + \cdots)t + (5 + 7 \cdot 11^2 + 10 \cdot 11^3 + 10 \cdot 11^4 + \cdots),$$

$$g_2(t) = t^2 + (3 + 5 \cdot 11 + 5 \cdot 11^2 + 2 \cdot 11^4 + \cdots)t + (5 + 11^3 + 7 \cdot 11^4 + \cdots),$$

$$g_3(t) = t - (7 + 2 \cdot 11 + 2 \cdot 11^2 + 10 \cdot 11^3 + 7 \cdot 11^4 + \cdots).$$

Let $g_1(\theta_1) = 0$, $g_2(\theta_2) = 0$, $g_3(\theta_3) = 0$. Following the notation of the beginning of §5 of [34] we denote the \mathbb{Q}_{11} -conjugates of the θ_i ’s as follows:

- $\theta_1^{(i)}$, $i = 1, 2$; the roots of $g_1(t)$, living in a quadratic extension of \mathbb{Q}_{11} .
- $\theta_2^{(i)}$, $i = 1, 2$; the roots of $g_2(t)$, living in a quadratic extension of \mathbb{Q}_{11} .
- $\theta_3^{(1)} = \theta_3 = 7 + 2 \cdot 11 + 2 \cdot 11^2 + 10 \cdot 11^3 + 7 \cdot 11^4 + \cdots \in \mathbb{Q}_{11}$; the root of $g_3(t)$.

Table 3: Application of “Second Corollary of Lemma 1” in [34] when $p = 11$

Notations in [34, §§ 3,5]	Corresponding values in this case and [references in this paper]
p	11
$g(t)$ (§3)	$t^5 + 65t^4 - 870t^3 - 18990t^2 + 26325t + 255069$ [(3.6)]
m (§3)	3; $g(t) = g_1(t)g_2(t)g_3(t)$ [begining of Subsection 3.2.3]
<i>continued on next page</i>	

continued from previous page	
Notations in [34, §§ 3,5]	Interpretations and [references] in this paper
\mathfrak{p}_1, e_1, d_1 \mathfrak{p}_2, e_2, d_2 (§3) \mathfrak{p}_3, e_3, d_3	$\langle \pi_{111} \rangle, 2, 1$ $\langle \pi_{112} \rangle, 2, 1$ $\langle \pi_{113} \rangle, 1, 1$ <div style="text-align: right;">[page 11]</div>
e (§5)	1
D_θ (§5)	$2^{32}3^{12}5^{11}11^6$ [(3.6)]

Since we intend to apply the *Prime Ideal Removing Lemma* [34, Lemma 1], we must compute

$$\max\{e_i, e_j\} \cdot \text{ord}_{11}(\theta_i^{(k)} - \theta_j^{(l)}), \quad i, j \in \{1, 2, 3\}, i \neq j, \quad (3.11)$$

where $k = 1$ if $i = 3$ and $k \in \{1, 2\}$ if $i = 1$ or 2 ; and analogously, $l = 1$ if $j = 3$ and $l \in \{1, 2\}$ if $j = 1$ or 2 . According to the discussion in Appendix A, in order to compute $\text{ord}_{11}(\theta_i^{(k)} - \theta_j^{(l)})$ for fixed $1 \leq i < j \leq 3$, it suffices to compute a polynomial $h_{ij}(t) \in \mathbb{Q}_{11}[t]$ such that $h_{ij}(\theta_j - \theta_i) = 0$. Moreover, since $\text{ord}_p(-\alpha) = \text{ord}_p(\alpha)$, it is clear that it suffices to consider only the values $(i, j) = (1, 3), (2, 3), (1, 2)$.

Obviously, $h_{i3}(t) = g_i(t + \theta_3)$, hence

$$h_{13}(t) = t^2 + (6 + 8 \cdot 11 + 11^2 + 8 \cdot 11^3 + 10 \cdot 11^4 + \dots)t + (9 + 6 \cdot 11 + 2 \cdot 11^2 + 8 \cdot 11^3 + \dots).$$

$$h_{23}(t) = t^2 + (6 + 10 \cdot 11 + 9 \cdot 11^2 + 9 \cdot 11^3 + 6 \cdot 11^4 + \dots)t + (9 + 9 \cdot 11 + 11^2 + 9 \cdot 11^3 + 9 \cdot 11^4 + \dots),$$

It follows that, when in (3.11) we have $(i, j) = (1, 3), (2, 3)$, then $\text{ord}_{11}(\theta_i^{(k)} - \theta_j^{(l)}) = 0$.

When $(i, j) = (1, 2)$, the following lemma, whose proof is a matter of straightforward calculations, gives us a quartic polynomial $h_{12}(t) \in \mathbb{Q}_{11}[t]$ which has $\theta_1^{(k)} - \theta_2^{(l)}$ as a zero (independent from k, l).

Lemma 3.1. *If $\theta_i^2 + a_i\theta_i + b_i = 0$ for $i = 1, 2$, then $\theta_2 - \theta_1$ is a root of $t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$, where*

$$c_3 = 2(a_2 - a_1), \quad c_2 = a_2^2 + a_1^2 - 3a_1a_2 + 2b_2 + 2b_1$$

$$c_1 = a_1^2a_2 - a_2^2a_1 - 2b_2a_1 - 2a_1b_1 + 2a_2b_2 + 2b_1a_2,$$

$$c_0 = b_2^2 + b_1^2 + b_2a_1^2 + b_1a_2^2 - b_2a_1a_2 - b_1a_1a_2 - 2b_2b_1.$$

The constant term of $h_{12}(t)$ is $9 \cdot 11^2 + 5 \cdot 11^3 + 6 \cdot 11^4 + \dots$, hence (A.1) gives $\text{ord}_{11}(\theta_1^{(k)} - \theta_2^{(l)}) = 2/4 = 1/2$.

In view of the above, when $(i, j) = (1, 3)$ or $(2, 3)$, the number (3.11) is zero, therefore, statement (i) of the aforementioned Prime Ideal Removing Lemma implies that $x - y\theta$ is divisible by at most one

prime among π_{111} and π_{113} (equivalently: $w_1 = 0$ or $w_3 = 0$) and by at most one prime among π_{112} and π_{113} (equivalently: $w_2 = 0$ or $w_3 = 0$), hence

$$\text{either } w_3 = 0 \text{ or } (w_1, w_2) = (0, 0). \quad (3.12)$$

When $(i, j) = (1, 2)$, the number (3.11) is equal to 1, hence, again by statement (i) of the Prime Ideal Removing Lemma, it follows that at most one among π_{111} and π_{112} divides $x - y\theta$ with power > 1 . *If this actually occurs* for π_{11i} ($i = 1$ or 2), which means that $\text{ord}_{\pi_{11i}}(x - y\theta) > 1$, then statement (ii) of the Prime Ideal Removing Lemma implies that

$$\text{ord}_{\pi_{11i}}(x - y\theta) \leq 2 \text{ord}_{11}(\theta_i^{(1)} - \theta_i^{(2)}) = 2 \cdot \frac{1}{2} = 1,$$

because, $(\theta_i^{(1)} - \theta_i^{(2)})^2$ being the discriminant of the polynomial $g_i(t)$, is equal to either $6 \cdot 11 + 8 \cdot 11^2 + O(11^3)$ if $i = 1$, or to $7 \cdot 11 + 2 \cdot 11^2 + O(11^3)$ if $i = 2$. This contradiction shows that $\text{ord}_{\pi_{11i}}(x - y\theta) \leq 1$ for both $i = 1, 2$, i.e.

$$w_i \leq 1 \quad (i = 1, 2). \quad (3.13)$$

If we combine (3.12) and (3.13) we see that we have the following possibilities:

$$(w_1, w_2, w_3) = (0, 0, w_3), (0, 1, 0), (1, 0, 0), (1, 1, 0), \quad (3.14)$$

where in the first case we understand that w_3 can be “large”. The remaining three possibilities combined with the relations (3.7) and (3.8), lead us to

$$\begin{aligned} \langle x - y\theta \rangle &= \langle \pi_2 \rangle^5 \langle \pi_{31} \rangle^4 \langle \pi_5 \rangle^{z_1} \langle \pi_{111} \rangle^{w_1} \langle \pi_{112} \rangle^{w_2} \\ \langle x - y\theta \rangle &= \langle \pi_2 \rangle^5 \langle \pi_{32} \rangle \langle \pi_5 \rangle^{z_1} \langle \pi_{111} \rangle^{w_1} \langle \pi_{112} \rangle^{w_2}. \end{aligned}$$

By (3.9), $z_2 = w_1 + w_2 + w_3 = w_1 + w_2 = 1, 2$, and by (3.10), $0 \leq z_1 \leq 27$. Taking norms in the above relations, we obtain the following fifty six Thue equations (cf. (3.5):

$$\begin{aligned} x^5 + 65x^4y - 870x^3y^2 - 18990x^2y^3 + 26325xy^4 + 255069y^5 &= c \\ c \in \{-2^5 \cdot 3^4 \cdot 5^{z_1} \cdot 11^{z_2} : 0 \leq z_1 \leq 27, 1 \leq z_2 \leq 2\}. \end{aligned} \quad (3.15)$$

The MAGMA routine for solving Thue equations, based on Bilu & Hanrot method [7] (which improves the method of [33]) “answers” that there are no solutions at all. The computation cost for this task is less than 2.5 seconds.

In view of the above discussion, we are left with the first case in (3.14), hence we have to solve the ideal equations $\langle x - y\theta \rangle = \langle \pi_2 \rangle^5 \langle \pi_{31} \rangle^4 \langle \pi_5 \rangle^{z_1} \langle \pi_{113} \rangle^{w_3}$ and $\langle x - y\theta \rangle = \langle \pi_2 \rangle^5 \langle \pi_{32} \rangle \langle \pi_5 \rangle^{z_1} \langle \pi_{113} \rangle^{w_3}$, where, in both cases, $0 \leq z_1 \leq 27$.

To sum up, the solution of the equation (3.5) is reduced to that of the equation

$$x - y\theta = \alpha \epsilon_1^{a_1} \epsilon_2^{a_2} \epsilon_3^{a_3} \epsilon_4^{a_4} \pi_{113}^{n_1} \quad (3.16)$$

$$\alpha \in \{\pi_2^5 \pi_{31}^4 \pi_5^{z_1}, \pi_2^5 \pi_{32} \pi_5^{z_1} : 0 \leq z_1 \leq 27\}, \quad n_1 = w_3 = z_2 = (b-1)/2.$$

in the unknowns $(a_1, a_2, a_3, a_4, n_1) \in \mathbb{Z}^4 \times \mathbb{Z}_{\geq 0}$.

3.2.4 From equation (3.16) to S -unit equation (3.17)

Let K be an extension of F such that $g(t)$ has at least three linear factors in $K[t]$. Actually, in our case, such an extension coincides with the splitting field of $g(t)$ over F (see (3.6)). We have $K = \mathbb{Q}(\omega)$ and the minimal polynomial of ω over \mathbb{Q} , denoted by $G(t)$, is of degree 20 (see Appendix B). Thus, there exist $\theta^{(i)}(t) \in \mathbb{Q}[t]$ ($i = 1, \dots, 5$), so that the \mathbb{Q} -conjugates $\theta^{(i)}$ of θ are

$$\theta^{(i)}(\omega) \in \mathbb{Q}(\omega) = K \quad (i = 1, \dots, 5).$$

For every $i \in \{1, \dots, 5\}$, the i -th embedding $F \hookrightarrow K$ is characterized by $\theta \mapsto \theta^{(i)}(\omega)$ and maps the general element $\beta \in F$ to its i -th conjugate $\beta^{(i)}(\omega)$. This belongs to $\mathbb{Q}(\omega)$, hence it is a polynomial expression in ω , of degree at most 19, with rational coefficients.

On the other hand, if \mathfrak{P} is the prime ideal of K over $\langle \pi_{113} \rangle$, mentioned in Appendix B,⁶ then, by the discussion of Appendix A, there is an embedding $K \hookrightarrow K_{\mathfrak{P}} = \mathbb{Q}_{11}(\omega_{\mathfrak{P}})$, where $G_{\mathfrak{P}}(\omega_{\mathfrak{P}}) = 0$ for a specific second-degree factor $G_{\mathfrak{P}}(t)$ of $G(t)$, irreducible over \mathbb{Q}_{11} ; see Appendix B. This embedding is characterized by $\omega \mapsto \omega_{\mathfrak{P}}$, so that the 11-adic roots of $g(t)$ are

$$\theta^{(i)}(\omega_{\mathfrak{P}}) \in \mathbb{Q}(\omega_{\mathfrak{P}}) = K_{\mathfrak{P}} \quad (i = 1, \dots, 5)$$

and, for every $\beta \in F$, if the i -th conjugate of β over \mathbb{Q} is $\beta^{(i)}(\omega)$ (see a few lines above), then the embedding $\omega \mapsto \omega_{\mathfrak{P}}$ maps β to $\beta^{(i)}(\omega_{\mathfrak{P}})$.

If we work p -adically with $p = 11$, then, by $\theta^{(i)}, \beta^{(i)}, \dots$ we will understand $\theta^{(i)}(\omega_{\mathfrak{P}}), \beta^{(i)}(\omega_{\mathfrak{P}}), \dots$; and if we work p -adically with $p = \text{infinite prime}$, by $\theta^{(i)}, \beta^{(i)}, \dots$ we will understand $\theta^{(i)}(\omega), \beta^{(i)}(\omega), \dots$. Our discussion below applies to both cases of p .

Applying the i -th embedding to the relation (3.16) we obtain the i -th conjugate relation

$$x - y\theta^{(i)} = \alpha^{(i)} \epsilon_1^{(i)a_1} \epsilon_2^{(i)a_2} \epsilon_3^{(i)a_3} \epsilon_4^{(i)a_4} \pi_{113}^{(i)n_1}.$$

⁶See just above and below of relation (B.1).

Then, for $i = i_0, j, k$, where $i_0, j, k \in \{1, \dots, 5\}$ are any three distinct indices, we obtain three conjugate relations, analogous to the above. Eliminating x, y from the these three relations we finally obtain (cf. [34, Section 7])

$$\lambda := \delta_1 \left(\frac{\pi_{113}^{(k)}}{\pi_{113}^{(j)}} \right)^{n_1} \prod_{i=1}^4 \left(\frac{\epsilon_i^{(k)}}{\epsilon_i^{(j)}} \right)^{a_i} - 1 = \delta_2 \left(\frac{\pi_{113}^{(i_0)}}{\pi_{113}^{(j)}} \right)^{n_1} \prod_{i=1}^4 \left(\frac{\epsilon_i^{(i_0)}}{\epsilon_i^{(j)}} \right)^{a_i}, \quad (3.17)$$

where

$$\delta_1 = \frac{\theta^{(i_0)} - \theta^{(j)}}{\theta^{(i_0)} - \theta^{(k)}} \cdot \frac{\alpha^{(k)}}{\alpha^{(j)}}, \quad \delta_2 = \frac{\theta^{(j)} - \theta^{(k)}}{\theta^{(k)} - \theta^{(i_0)}} \cdot \frac{\alpha^{(i_0)}}{\alpha^{(j)}}. \quad (3.18)$$

Now and until the end of the paper we put

$$H = \max\{n_1, |a_1|, |a_2|, |a_3|, |a_4|\}$$

3.2.5 Equation (3.17) implies an upper bound $n_1 \leq c_{13} \log H$

We will prove the inequality in its title of this subsection, where c_{13} is given by (3.20). Our main tool is the important Theorem 3.2 due to Kunrui Yu which, given the algebraic numbers $\alpha_1, \dots, \alpha_n$ and a prime p , provides an upper bound for the p -adic valuation of $\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1$, for any $b_1, \dots, b_n \in \mathbb{Z}$, in terms of $\log \max\{3, |b_1|, \dots, |b_n|\}$.

We turn to the relation (3.17), which we view as an algebraic relation over \mathbb{Q}_{11} . According to the discussion in Appendix B, the 11-adic roots $\theta^{(i)} \in \mathbb{C}_{11}$ of $g(t)$ are identified with $\theta_i(\omega_{\mathfrak{P}})$ ($i = 1, \dots, 5$).

We choose the indices i_0, j, k following the instructions in [34], bottom of p. 235 and beginning of p. 236 up to Lemma 3. According to the discussion therein, since π_{113} corresponds to the polynomial $g_3(t)$ whose root is $\theta_5(\omega_{\mathfrak{P}})$ (cf. end of Appendix B), we must choose $i_0 = 5$; and since $\theta_1(\omega_{\mathfrak{P}})$ and $\theta_3(\omega_{\mathfrak{P}})$ are (according to the end of Appendix B, again) roots of the quadratic irreducible polynomial $g_1(t) \in \mathbb{Q}_{11}[t]$, we can choose $j = 1$ and $k = 3$. In view of [34, Lemma 3 (i)], $\text{ord}_{11}(\pi_{113}^{(k)}/\pi_{113}^{(j)}) = 0$ and by [34, Corollary of Lemma 2 (i)], $\text{ord}_{11}(\epsilon_i^{(k)}/\epsilon_i^{(j)}) = 0$ for $i = 1, \dots, 4$. Also, since $\theta^{(k)}, \theta^{(j)}$ are 11-adic roots of a second degree irreducible polynomial over \mathbb{Q}_{11} , it follows, according to the second “bullet” in page 236 of [34], that $\text{ord}_{11}(\delta_1) = 0$. These facts will be used in the application of Theorem 3.2.

Also, the relation (13) of [34, Theorem 5] holds, which in our case reads $\text{ord}_{11}(\lambda) = \text{ord}_{11}(\delta_2) + n_1$ ⁷. A computation shows that $\text{ord}_{11}(\delta_2) = 1/2$ ⁸, hence

$$\text{ord}_{11}(\lambda) = n_1 + \frac{1}{2} \quad (3.19)$$

⁷Actually, according to the relation (13) of [34, Theorem 5], n_1 is multiplied by a positive integer h_1 , defined in [34, Section 6], which is a divisor of the order of the ideal-class group. In our case, the ideal-class group is trivial, hence $h_1 = 1$.

⁸By (3.18) and (3.16) there are 56 possible values for δ_2 .

Now we are ready to apply Theorem 3.2. With four minor corrections, this is Theorem 11.1 of K. Hambrook's thesis [18]. It is a consequence of Theorems 1 and 3 of [37] and the Lemma in the Appendix of [36].

Theorem 3.2 (Kunrui Yu). *Let $\alpha_1, \dots, \alpha_n$ ($n \geq 2$) be non-zero algebraic numbers and*

$$K = \mathbb{Q}(\alpha_1, \dots, \alpha_n), \quad D = [K : \mathbb{Q}].$$

Let p be a rational prime, \mathfrak{P} a prime ideal of the ring of integers of K lying above p and $e_{\mathfrak{P}} = e_{K/\mathbb{Q}}(\mathfrak{P})$, $f_{\mathfrak{P}} = f_{K/\mathbb{Q}}(\mathfrak{P})$ the ramification index and residue class degree, respectively, of \mathfrak{P} .

Now define d and f as follows:

If $p = 2$ then

$$d = \begin{cases} D & \text{if } e^{2\pi/3} \in K \\ 2D & \text{if } e^{2\pi/3} \notin K \end{cases}, \quad f = \begin{cases} f_{\mathfrak{P}} & \text{if } e^{2\pi/3} \in K \\ \max\{2, f_{\mathfrak{P}}\} & \text{if } e^{2\pi/3} \notin K \end{cases}.$$

If $p \geq 3$ and $p^{f_{\mathfrak{P}}} \equiv 3 \pmod{4}$ then

$$d = D, \quad f = f_{\mathfrak{P}}.$$

If $p \geq 3$ and $p^{f_{\mathfrak{P}}} \equiv 1 \pmod{4}$ then

$$d = \begin{cases} D & \text{if } e^{2\pi/4} \in K \\ 2D & \text{if } e^{2\pi/4} \notin K \end{cases}, \quad f = \begin{cases} f_{\mathfrak{P}} & \text{if } e^{2\pi/4} \in K \text{ or } p \equiv 1 \pmod{4} \\ \max\{2, f_{\mathfrak{P}}\} & \text{if } e^{2\pi/4} \notin K \text{ and } p \equiv 3 \pmod{4} \end{cases}.$$

Put

$$\tau = \frac{p-1}{p-2}, \quad \kappa = \left\lceil \log \left(\frac{2e_{\mathfrak{P}}}{p-1} \right) / \log p \right\rceil, \quad Q = \begin{cases} 3 & \text{if } p = 2 \\ 4 & \text{if } p \geq 3 \text{ and } p^f \equiv 1 \pmod{4} \\ 1 & \text{if } p \geq 3 \text{ and } p^f \equiv 3 \pmod{4} \end{cases},$$

$$(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6) = \begin{cases} (160, 32, 40, 276, 16, 8) & \text{if } p = 2 \\ (759, 16, 20, 1074, 8, 4) & \text{if } p = 3, d \geq 2 \\ (537, 16, 20, 532, 8, 4) & \text{if } p = 3, d = 1 \\ (1473, 8\tau, 10, 394\tau, 8, 4) & \text{if } p \geq 5, e_{\mathfrak{P}} = 1, p \equiv 1 \pmod{4} \\ (1282, 8\tau, 10, 366\tau, 8, 4) & \text{if } p \geq 5, e_{\mathfrak{P}} = 1, p \equiv 3 \pmod{4}, d \geq 2 \\ (1288, 8\tau, 10, 396\tau, 8, 4) & \text{if } p \geq 5, e_{\mathfrak{P}} = 1, p \equiv 3 \pmod{4}, d = 1 \\ (319, 16, 20, 402, 8, 4) & \text{if } p = 5, e_{\mathfrak{P}} \geq 2 \\ (1502, 16, 20, 1372, 8, 4) & \text{if } p \geq 7, e_{\mathfrak{P}} \geq 2, p \equiv 1 \pmod{4} \\ (2190, 16, 20, 1890, 8, 4) & \text{if } p \geq 7, e_{\mathfrak{P}} \geq 2, p \equiv 3 \pmod{4} \end{cases}$$

$$\begin{aligned}
c_2 &= \frac{(n+1)^{n+2} d^{n+2}}{(n-1)!} \frac{p^f}{(f \log p)^3} \max\{1, \log d\} \max\{\log(e^4(n+1)d), e_{\mathfrak{P}}, f \log p\}, \\
c'_3 &= \kappa_1 \kappa_2^n \left(\frac{n}{f \log p} \right)^{n-1} \prod_{i=1}^n \max \left\{ h(\alpha_i), \frac{f \log p}{\kappa_3(n+4)d} \right\}, \\
c''_3 &= \kappa_4 (e \kappa_5)^n p^{(n-1)\kappa} \prod_{i=1}^n \max \left\{ h(\alpha_i), \frac{1}{e^2 \kappa_6 p^\kappa d} \right\}.
\end{aligned}$$

Let b_1, \dots, b_n be rational integers and define

$$\lambda = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1, \quad B = \max\{3, |b_1|, \dots, |b_n|\}.$$

If $\lambda \neq 0$ and $\text{ord}_{\mathfrak{P}}(\alpha_i) = 0$ for $i = 1, \dots, n$, then

$$\text{ord}_p(\lambda) < c'_{10} \log B,^9 \quad c'_{10} = \frac{c_2 \min\{c'_3, c''_3\}}{Q \cdot e_{\mathfrak{P}}}.$$

Now we apply Theorem 3.2 to the λ given in (3.17), as interpreted in the beginning of this section, with $i_0 = 5, j = 1, k = 3$. Our application is briefly described in Table 4.

Table 4: Application of Theorem 3.2

Notations in Theorem 3.2	Values in this paper
n	6
(α_1, b_1)	$(\delta_1, 1)$ [equation (3.18) with $(i_0, j, k) = (5, 1, 3)$]
(α_2, b_2)	$(\pi_{113}^{(3)}/\pi_{113}^{(1)}, n_1)$ [equation (3.17)]
$(\alpha_i, b_i), (i = 3, 4, 5, 6)$	$(\epsilon_i^{(3)}/\epsilon_i^{(1)}, a_{i-2}), (i = 3, 4, 5, 6)$ [equation (3.17)]
K	K [Appendix B]
D	20
p	11
\mathfrak{P}	\mathfrak{P} [Appendix B just above equation (B.1)]
$(f_{\mathfrak{P}}, e_{\mathfrak{P}}) = (f_{K/\mathbb{Q}}(\mathfrak{P}), e_{K/\mathbb{Q}}(\mathfrak{P}))$	$(1, 2); [(B.1)]$
<i>continued on next page</i>	

⁹We use the notation c'_{10} in order to conform with the notation of [34, page 238].

<i>continued from previous page</i>	
Notations in Theorem 3.2	Values in this paper
B	$\max\{3, n_1, a_1 , a_2 , a_3 , a_4 \}$
c'_{10}	$< 9.9 \cdot 10^{30}$

A remark has its place here: By (3.16) and the definition of δ_1 in (3.18) we see that δ_1 runs through a set of cardinality 28, therefore, for each value of δ_1 , we must compute the parameter c'_{10} . It turns out that, in all cases, $c'_{10} < 9.9 \cdot 10^{30}$ and this is mentioned in the above table. Also, in the notation of [34], the use of Theorem 3.2 always implies $c'_{11} = 0$.

By writing a number of rather simple routines we automated the computations. Finally, by setting $c'_{10} \leftarrow \max_{\delta_1} c'_{10}$ and $c'_{11} \leftarrow \max_{\delta_1} c'_{11}$ we find $c'_{10} = 9.99 \cdot 10^{30}$ and $c'_{11} = 0$.

By [34, relation (14)], $n_1 \leq c_{13}(\log H + c_{14})$, where c_{13}, c_{14} are explicitly computed from c'_{10} and c'_{11} following the simple instructions found on p. 238 of [34]. The difference between the pairs (c_{13}, c_{14}) and (c'_{10}, c'_{11}) , if any at all, is negligible in practice. Anyway, in our case, it turns out easily that the two pairs coincide and, therefore,

$$n_1 \leq c_{13}(\log H + c_{14}), \quad c_{13} = 9.99 \cdot 10^{30}, \quad c_{14} = 0, \quad (3.20)$$

where

$$H = \max\{n_1, |a_1|, |a_2|, |a_3|, |a_4|\}. \quad (3.21)$$

A computational remark. According to the instruction of [34, page 238], in order to compute c_{13} from c'_{10} we need the least positive integer h such that \mathfrak{p}^h is principal. In our case \mathfrak{p} is already a principal ideal, therefore we take $h = 1$. In order to compute c_{14} from c'_{11} we need to compute $\text{ord}_{11}(\delta_2)$ for the 56 values δ_2 (cf. (3.17) and (3.18)). One shouldn't expect difficulties in carrying out such computations using MAGMA or any other package specialized to Number Theory.

3.2.6 First explicit bounds for $H = \max\{n_1, |a_1|, |a_2|, |a_3|, |a_4|\}$ and n_1

We will prove the numerical upper bound (3.23) for H , based to E.M. Matveev's lower bound for linear form in (real/complex) logarithms of algebraic numbers; see Theorem 3.3 below. Then, as a straightforward consequence of (3.20), this will imply the numerical upper bound (3.24) for n_1 .

We focus our attention to $1 + \lambda$, where λ is defined in relation (3.17). In this section we view K embedded in the complex field \mathbb{C} , so that the algebraic numbers appearing in λ are complex numbers; actually, they are all *real* numbers, because all roots of $g(t)$ are real. Note that the indices i_0, j, k

figuring in (3.17) are *any* distinct indices from the set $\{1, \dots, 5\}$. We follow step by step the very explicit instructions of Sections 9 and 10 of [34] in order to compute a chain of constants (in the order that are displayed below)

$$c_{15}, c_{16} = 0.129, {}^{10} c'_{17}, c'_{18}, c''_{17}, c''_{18}, c_{17}, c_{18}, c_{19}, c_{20}, c_{12}, c_{21}, c_{22}.$$

This is a rather boring and cumbersome task if one performs the computations “by hand” (with the aid of a pocket calculator). Fortunately, the instructions are programmable in MAGMA without much difficulty, so that the chain of computations is performed automatically. It turns out that $c_{22} = 14$. According to the terminology of page 243 of [34], we are treating a “real case”. Moreover, by page 244 of [34], if we assume that $H > c_{22} = 14$ (H is defined in (3.21)), then $1 + \lambda$ is a positive real number and

$$\Lambda = \log(1 + \lambda) = \log |\delta_1| + n_1 \log \left| \frac{\pi_{113}^{(k)}}{\pi_{113}^{(j)}} \right| + \sum_{i=1}^4 a_i \log \left| \frac{\alpha_{113}^{(k)}}{\alpha_{113}^{(j)}} \right|. \quad (3.22)$$

By a strong and handy result of E.M. Matveev we can compute explicit constants c_7, c_8 such that $\log(1 + \lambda) > \exp(-c_7(\log H + c_8))$. More specifically we have the Theorem 3.3 below, which is a slight restatement of Theorem 2.1 of [24]. In this theorem \log denotes an arbitrary but fixed branch of the logarithmic function on \mathbb{C} ; if x is a positive real number, $\log x$ always means real (natural) logarithm of x .

Theorem 3.3. ([24, Theorem 2.1]) *Let $\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_n$, where $b_1, \dots, b_n \in \mathbb{Z}$ with $b_n \neq 0$, and $\alpha_1, \dots, \alpha_n$ are algebraic numbers of degree at most D , embedded in \mathbb{C} , and $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over \mathbb{Z} .*

Consider A_1, \dots, A_n satisfying

$$A_i \geq \max\{Dh(\alpha_i), |\log \alpha_i|\} \quad 1 \leq i \leq n,$$

where, in general, $h(\alpha)$ denotes the absolute logarithmic height of the algebraic number α . Set $\kappa = 1$ if all α_i 's are real; otherwise set $\kappa = 2$. Next, define

$$A = \max_{1 \leq i \leq n} A_i / A_n, \quad \Omega = A_1 \cdots A_n$$

and

$$B = \max_{1 \leq i \leq n} |b_i|.$$

Then

$$|\Lambda| > \exp(-c_7(\log B + c_8)),$$

¹⁰We give the value of c_{16} , because this will play a role later.

where

$$c_7 = \frac{16}{n!\kappa} e^n (2n+1+2\kappa)(n+2)(4(n+1))^{n+1} (en/2)^\kappa \log(e^{4.4n+7} n^{5.5} D^2 \log(eD)) D^2 \Omega,$$

$$c_8 = \log(1.5eD \log(eD)A).$$

Now we apply Theorem 3.3 to the linear form $\Lambda = \log(1+\lambda)$ in (3.22). Following the instructions of [34] (bottom of page 249 - beginning of page 250), we must consider Λ for all $i_0 \in \{1, \dots, 5\}$, and for each specific i_0 , the choice of the indices j, k is arbitrary, provided that $i_0 \neq j \neq k \neq i_0$. Note that the condition of \mathbb{Z} -linear independence of the α_i 's, imposed by Theorem 3.3, in our case reads $\log(1+\lambda) \neq 0$. This is equivalent to $\lambda \neq 0$; we see that this is true by viewing λ as the right-hand side of the relation (3.17). The application of Theorem 3.3 in our case is briefly described in Table 5. In this table, (i_0, j, k) runs through the set $\{(1, 2, 3), (2, 1, 3), (3, 1, 2), (4, 1, 2), (5, 1, 2)\}$. We note that, the condition of \mathbb{Z} -linear independence of the α_i 's, imposed by Theorem 3.3, in our case reads $\log(1+\lambda) \neq 0$. This is equivalent to $\lambda \neq 0$; we see that this is true by viewing λ as the right-hand side of the relation (3.17).

Table 5: Application of Theorem 3.3 to $\log(1+\lambda)$

Notations in Theorem 3.3	Values in this paper
n	6
(α_1, b_1)	$(\delta_1, 1)$ [equation (3.18) with $(i_0, j, k) = (5, 1, 3)$]
(α_2, b_2)	$(\pi_{113}^{(3)}/\pi_{113}^{(1)}, n_1)$ [equation (3.17)]
$(\alpha_i, b_i), (i = 3, 4, 5, 6)$	$(\epsilon_i^{(3)}/\epsilon_i^{(1)}, a_{i-2}), (i = 3, 4, 5, 6)$ [equation (3.17)]
K	K [Appendix B]
D	20
p	11
\mathfrak{P}	\mathfrak{P} [Appendix B just above equation (B.1)]
$(f_{\mathfrak{P}}, e_{\mathfrak{P}}) = (f_{K/\mathbb{Q}}(\mathfrak{P}), e_{K/\mathbb{Q}}(\mathfrak{P}))$	$(1, 2); [(B.1)]$
B	$\max\{3, n_1, a_1 , a_2 , a_3 , a_4 \}$
c_7, c_8	$< 4.8626 \cdot 10^{27}, < 5.7864$

We remark at this point that, actually, the values of c_7, c_8 which we obtain for the various choices of (i_0, j, k) differ “very little”, if they differ at all.

We¹¹ continue to follow the instructions from the relation (24) of [34] onwards and compute constants c_{23}, c_{24}, c_{25} and, finally, constants c_{real} and c_{27} , such that

- $H = \max\{n_1, |a_1|, |a_2|, |a_3|, |a_4|\} < c_{\text{real}}$ (see [34, Theorem 10]),

¹¹ “We” means “our MAGMA code”.

- $|\Lambda| \leq c_{27} \exp\{-c_{16} \max_{1 \leq i \leq 4} |a_i|\}$ (see [34, relation (29)]).

Also, by [34, Corollary of Theorem 10], $n_1 \leq c_{13}(\log c_{\text{real}} + c_{14})$ and, in general, n_1 is considerably smaller than c_{real} .

According to our computations, the maximum value for c_{real} ¹² is $1.3216 \cdots \times 10^{43}$, from which we conclude that

$$H < 1.3217 \cdot 10^{43} =: K_0 \quad (3.23)$$

and

$$n_1 < 9.918312 \cdot 10^{32} =: N_0. \quad (3.24)$$

Also, $c_{27} < 3.906653$; this constant, along with $c_{16} = 0.129$ will be used in Subsection 3.2.7.

Computation time. The totality of computations that led to the bounds (3.23) and (3.24) was about 8 minutes.

3.2.7 The first p -adic reduction

In this section we reduce the upper bound (3.24) by a process we call *p-adic reduction*, with $p = 11$ in our case. For the basic facts we refer to [35] and [34, Sections 12,14,15]. Given a rational prime p and a p -adic number x (in general, x belongs to a finite extension of \mathbb{Q}_p), the p -adic logarithm of x is denoted by $\log_p x$ and belongs to the same extension of \mathbb{Q}_p in which x belongs.

We go back to the relations (3.17) and (3.18). According to the discussion in Appendix B and, more specifically, the notations etc on page 31, we have an embedding $K \hookrightarrow K_{\mathfrak{P}}$, where $K_{\mathfrak{P}} = \mathbb{Q}_{11}(\omega_{\mathfrak{P}})$ is a quadratic extension of \mathbb{Q}_{11} defined by the polynomial $G_{\mathfrak{P}}(t) = t^2 + (10744341441 + O(11^{10}))t + (9625552201 + O(11^{10})) = 0$,¹³ which allows us to view the $\theta^{(i)}$'s figuring in the above relations as elements of $K_{\mathfrak{P}}$. According to our choice for i_0, j, k , made in page 16, $(i_0, j, k) = (5, 1, 3)$ and, consequently, $\theta^{(i_0)} = \theta^{(5)} = 7050162550 + O(11^{10})$, $\theta^{(j)} = \theta^{(1)} = (9038034724 + O(11^{10}))\omega_{\mathfrak{P}} + (8245826831 + O(11^{10}))$, $\theta^{(k)} = \theta^{(3)} = (4757114675 + O(11^{10}))\omega_{\mathfrak{P}} + (5113588460 + O(11^{10}))$. In the notation of page 16, these are the 11-adic roots $\theta_5(\omega_{\mathfrak{P}})$, $\theta_1(\omega_{\mathfrak{P}})$ and $\theta_3(\omega_{\mathfrak{P}})$ of $g(t)$, respectively¹⁴. By (3.19), $\text{ord}_{11}(\lambda) = n_1 + \frac{1}{2}$, therefore, by [34, Lemma 12], $\text{ord}_{11}(\Lambda_{11}) = n_1 + \frac{1}{2}$, where

$$\Lambda_{11} = \log_{11} \delta_1 + n_1 \log_{11} \left(\frac{\pi_{113}^{(3)}}{\pi_{113}^{(1)}} \right) + \sum_{i=1}^4 a_i \log_{11} \left(\frac{\epsilon_i^{(3)}}{\epsilon_i^{(1)}} \right).$$

¹²As (i_0, j, k) runs through the set $\{(1, 2, 3), (2, 1, 3), (3, 1, 2), (4, 1, 2), (5, 1, 2)\}$.

¹³see Table 6 et. seq.

¹⁴The two remaining 11-adic roots of $g(t)$ are $\theta_2(\omega_{\mathfrak{P}}) = (517324682 + O(11^{10}))\omega_{\mathfrak{P}} + (5431351847 + O(11^{10}))$ and $\theta_4(\omega_{\mathfrak{P}}) = (2621442663 + O(11^{10}))\omega_{\mathfrak{P}} + (7443205770 + O(11^{10}))$.

Because δ_1 depends also on the choice of α (cf. relations (3.18) and (3.16)), there are 56 possibilities for Λ_{11} . Therefore, in what follows, we assume that, having chosen $\alpha \in \{\pi_2^5 \pi_{31}^4 \pi_5^{z_1}, \pi_2^5 \pi_{32} \pi_5^{z_1} : 0 \leq z_1 \leq 27\}$, we compute the 11-adic logarithms appearing in Λ_{11} ; except for $\log_{11} \delta_1$, the remaining logarithms are independent from α .

Note that the values of \log_{11} above belong to $K_{\mathfrak{P}}$ and, therefore, they are of the form $x_0 + x_1 \omega_{\mathfrak{P}}$, where $x_0, x_1 \in \mathbb{Q}_p$. If we put

$$\log_{11} \delta_1 = \rho_0 + \rho_1 \omega_{\mathfrak{P}}, \log_{11}(\pi_{113}^{(3)}/\pi_{113}^{(1)}) = \lambda_0 + \lambda_1 \omega_{\mathfrak{P}}, \log_{11}(\epsilon_i^{(3)}/\epsilon_i^{(1)}) = \mu_{i0} + \mu_{i1} \omega_{\mathfrak{P}}$$

($i = 1, 2, 3, 4$), then $\Lambda_{11} = \Lambda_{11,0} + \Lambda_{11,1} \omega_{\mathfrak{P}}$, where

$$\Lambda_{11,0} = \rho_0 + n_1 \lambda_0 + \sum_{i=1}^4 a_i \mu_{i0}, \quad \Lambda_{11,1} = \rho_1 + n_1 \lambda_1 + \sum_{i=1}^4 a_i \mu_{i1}.$$

Following the instructions of [34, p.p. 256-257] we put for $i = 0, 1$:

$$\Lambda'_{11,i} = -\beta_{0i} - n_1 \beta_{1i} - a_1 \beta_{2i} - a_3 \beta_{3i} + a_4,$$

where,

$$\beta_{0,i} = -\rho_i/\mu_{4i}, \quad \beta_{1i} = -\lambda_i/\mu_{4i}, \quad \beta_{ji} = -\mu_{j-1,i}/\mu_{4i} \quad (j = 2, 3, 4).$$

We divided by $\text{ord}_{11}(\mu_{4i})$, because $\text{ord}_{11}(\mu_{4i}) \leq \min\{\text{ord}_{11}(\lambda_1), \text{ord}_{11}(\mu_{1i}), \dots, \text{ord}_{11}(\mu_{4i})\}$.

Following the detailed instructions of [34, Section 15], we put $W = \lceil K_0/N_0 \rceil$, and we choose appropriately a number $\kappa > 1$ –this will become clear below– and an integer m such that

$$11^m W = \kappa K_0^5,^{15} \tag{3.25}$$

Then, for $i = 0, 1$, we consider the lattice Γ_{mi} which is generated by the column-vectors of the matrix

$$\begin{pmatrix} W & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \beta_{1i}^{(m)} & \beta_{2i}^{(m)} & \beta_{3i}^{(m)} & \beta_{4i}^{(m)} & 11^m \end{pmatrix},$$

where, in general, for $\beta \in \mathbb{Q}_{11}$, we denote by $\beta^{(m)}$ the integer of the interval $[0, 11^m - 1]$ for which $\text{ord}_{11}(\beta - \beta^{(m)}) \geq m$. We also consider the column vector

$$\mathbf{y}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\beta_{0i}^{(m)} \end{pmatrix}.$$

¹⁵The exponent 5 is equal to the number of the unknown exponents in (3.17).

Note that, in view of our remark after the definition of Λ_{11} , there are 56 possible values for the vector \mathbf{y}_i , but the lattices Γ_{mi} are independent from α .

Let $\mathbf{c}_1, \dots, \mathbf{c}_5$ be the column-vectors of an (ordered) LLL-reduced basis of Γ_{mi} and $s_1, \dots, s_5 \in \mathbb{Q}$ be such that $\mathbf{y}_i = \sum_{j=1}^5 s_j \mathbf{c}_j$. Let j_0 be the maximum index $j \in \{1, \dots, 5\}$ for which $s_j \notin \mathbb{Z}$ and denote by $\|s_{j_0}\|$ the distance of s_{j_0} from the nearest to it integer. Finally, put

$$\ell(\Gamma_{mi}, \mathbf{y}_i) \geq \begin{cases} \frac{1}{4} |\mathbf{c}_1| & \text{if } \mathbf{y}_i = \mathbf{0} \\ \frac{1}{4} \|s_{j_0}\| \cdot |\mathbf{c}_1| & \text{if } \mathbf{y}_i \neq \mathbf{0} \end{cases}$$

Now we apply [34, Proposition 15], which, in our case reads:

If

$$\ell(\Gamma_{mi}, \mathbf{y}_i) > \sqrt{WN_0^2 + 4K_0^2}, \quad (3.26)$$

*then $n_1 < m$.*¹⁶

Heuristically, when κ –and, accordingly by (3.25), also m – are sufficiently large, it is “reasonable” to expect that condition (3.26) is satisfied, which would imply an upper bound for n_1 . Choosing in (3.25) $\kappa = 100$, so that $m = 206$, we check that the condition (3.26) is satisfied, for either $i = 0$ or $i = 1$, for all but 10 values of \mathbf{y}_i ; for the ten exceptional values of \mathbf{y}_i we take $\kappa = 1000$, so that $m = 207$, and then (3.26) is satisfied.

As a consequence, we conclude that $n_1 \leq 207$.

Computation time. The computation cost for this reduction step was less than 1 minute.

3.2.8 The first reduction over \mathbb{R}

We have the upper bound $K_0 = 1.32171 \cdot 10^{43}$ for $H = \max\{n_1, |a_1|, |a_2|, |a_3|, |a_4|\}$ and, by the conclusion of Subsection 3.2.7, we already know that $n_1 \leq 207 =: N_1$. Thus, in (3.22), coefficients n_1, a_1, \dots, a_4 of the linear form $\log(1 + \lambda)$ satisfy $n_1 \leq 207$ and $\max_i |a_i| \leq K_0$. Referring to (3.22), let us put

$$\Lambda = \log(1 + \lambda) = \rho + n_1 \lambda_1 + \sum_{i=1}^4 a_i \mu_i,$$

where the meaning of the real numbers $\rho, \lambda_1, \mu_1, \dots, \mu_4$ is obvious. Once again we stress the fact that these six real numbers depend on the choice of the indices (i_0, j, k) (cf. page 21), and $\rho = \log |\delta_1|$

¹⁶Actually, in accordance to [34, Proposition 15], the upper bound for n_1 is $(m - l)/h$. By the fact that, over 11, the prime ideals of F are principal, and the definition of h in [34, page 234], we have $h = 1$. Also, l is a small number, explicitly determined in page 257 of [34] and, more specifically, below the relation (32); in our case, it turns out that $l = 0$.

(cf. (3.22)) depends also on the choice of α (cf. relations (3.18) and (3.16)). Since there are 5 choices for (i_0, j, k) and 56 choices for α , this implies that there are $5 \times 56 = 280$ possibilities for the linear form Λ . Therefore, in what follows, we assume that, having chosen $\alpha \in \{\pi_2^5 \pi_{31}^4 \pi_5^{z_1}, \pi_2^5 \pi_{32} \pi_5^{z_1} : 0 \leq z_1 \leq 27\}$ and $(i_0, j, k) \in \{(1, 2, 3), (2, 1, 3), (3, 1, 2), (4, 1, 2), (5, 1, 2)\}$, we compute the real numbers $\rho, \lambda_1, \mu_1, \dots, \mu_4$.

We follow the reduction process of [35], as presented in [34, Section 16]. We put $W' = \lceil K_0/N_1 \rceil$ –this is independent from the above choices– and choose a number $\kappa > 1$ and an integer C so that $CW' \approx \kappa K_0^5$.¹⁷ How we choose κ will become clear below; as it turns out in practice, κ depends on α and (i_0, j, k) . We consider the lattice Γ_C which is generated by the column-vectors of the matrix

$$\begin{pmatrix} W' & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \phi_1 & \psi_1 & \psi_2 & \psi_3 & \psi_4 \end{pmatrix},$$

where $\phi_1 = \lfloor C\lambda_1 \rfloor, \psi_i = \lfloor C\mu_i \rfloor$ ($i = 1, \dots, 4$). Also, we put $\phi_0 = \lfloor C\rho \rfloor$ and consider the column-vector

$$\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\phi_0 \end{pmatrix}.$$

As in the previous section, we compute an (ordered) LLL-reduced basis of Γ_C , say $\mathbf{c}_1, \dots, \mathbf{c}_5$. Let $s_1, \dots, s_5 \in \mathbb{Q}$ be the coefficients of \mathbf{y} with respect to this basis, denote by j_0 the maximum index $j \in \{1, \dots, 5\}$ for which $s_j \notin \mathbb{Z}$ and by $\|s_{j_0}\|$ the distance of s_{j_0} from the nearest to it integer. Finally, put

$$\ell(\Gamma_C, \mathbf{y}) \geq \begin{cases} \frac{1}{4}|\mathbf{c}_1| & \text{if } \mathbf{y} = \mathbf{0} \\ \frac{1}{4} \|s_{j_0}\| \cdot |\mathbf{c}_1| & \text{if } \mathbf{y} \neq \mathbf{0}. \end{cases}$$

Following the instructions of [34, page 265] we put $R = N_1 + 4K_0 + 1$ and $S = W'^2 N_1^2 + 3K_0^2$. By [34, Proposition 16]:

$$\text{If } \ell(\Gamma_C, \mathbf{y}) \geq \sqrt{R^2 + S} \quad (*)$$

$$\text{then } H \leq \frac{1}{c_{16}} \{\log c_{27} + \log C - \log(\sqrt{\ell(\Gamma_C, \mathbf{y})^2 - S} - R)\}. \quad (**)$$

¹⁷As in Subsection 3.2.7, the exponent 5 is the number of the unknowns exponents in (3.17).

Heuristically, one can argue that, if κ is sufficiently large and $C = \lceil \kappa K_0^5 / W' \rceil$, then it is “reasonable” to expect that $\ell(\Gamma_C, \mathbf{y}) \geq \sqrt{R^2 + S}$ and, consequently, an upper bound for H is obtained from (**), which is of the size of $\log K_0$.

To give an idea, if $(i_0, j, k) = (1, 2, 3)$ and we take $\kappa = 100$, $C = 10^{187}$, then the condition (*) is satisfied for all α ’s and, as α runs through all its possible values, the maximum bound (**) is 229. If $(i_0, j, k) = (2, 1, 3)$ and $\kappa = 100$, $C = 10^{187}$, then (*) holds for all but 11 values of α . For the 45 “successful” values of α the maximum bound (**) is 229. For the 11 remaining values the condition (*) holds if we take $\kappa = 500$ and $C = 10^{187}$; then the maximum upper bound (**) is 231.

In this way we finally obtain the upper bound $H \leq 231 =: K_1$, valid for all choices of α and (i_0, j, k) mentioned in the beginning of this section.

Computation time. At this stage, the computation time was less than half of a minute.

3.2.9 Further reduction and final stage of resolution

We repeat the p -adic reduction process of Subsection 3.2.7 with $K_0 \leftarrow K_1 = 231$ and $N_0 \leftarrow N_1 = 207$. This affects W and, consequently, κ and m in (3.25), which now becomes “very small”. Thus, we obtain the new bound $n_1 \leq N_2 = 25$, and this took less than 1 minute.

Next, applying the reduction process of Subsection 3.2.7 with $K_0 \leftarrow K_1 = 231$ and $N_1 \leftarrow N_2$, implies $H \leq K_2 = 41$; this took a few seconds.

A third p -adic reduction step can improve a little bit the upper bound for n_1 . The process of Subsection 3.2.7 with $K_0 \leftarrow K_2 = 41$ and $N_0 \leftarrow N_2 = 25$ implies $n_1 \leq N_3 = 21$, and this took around 1 second. Although we can make a further reduce to the bound of H , as well, and obtain $H \leq 34$, we will not use this.

Actually, we prefer to solve a set of Thue equations (3.15), with right-hand side $c \in \{-2^5 3^4 5^{z_1} 11^{z_2} : 0 \leq z_1 \leq 27, z_2 = 0 \text{ or } 3 \leq z_2 \leq 21\}$, using MAGMA’s implementation of Bilu & Hanrot’s method [7]. Remember that, as already mentioned a few lines below (3.15), no solutions exist when $c \in \{-2^5 3^4 5^{z_1} 11^{z_2} : 0 \leq z_1 \leq 27, 1 \leq z_2 \leq 2\}$. Thus, we are left with 28×20 Thue equations, using the above implementation. This is the most expensive task; it took us about 7662 secs $\approx 2\text{h } 7' 42''$. No solutions were found, hence, we have the following result:

Proposition 3.4. *There are no solutions to the equation (3.5), hence, by Subsection 3.1, equation (1.2) with abx odd and $n = 5$ has no solutions.* \square

Now, Propositions 2.1 and 3.4 complete the proof of Theorem 1.1.

Remark: Taking into account the computation cost at previous stages from Subsections 3.2.6 through 3.2.8, we see that the total computation time for the needs of Subsection 3.2 is less than 2h.30'.

Appendix A

Working p -adically. Some general facts.

In this appendix we combine several facts which are scattered in the literature. Our basic references are [8], [12], [14], [20], [25].

Let p be a rational prime. For every non-zero $x \in \mathbb{Q}$ we denote by $v_p(x)$ the exponent with which p appears in the prime factorization of x and, as usually, the p -adic absolute value of x is defined by $|x|_p = p^{-v_p(x)}$. We set, by convention, $v_p(0) = -\infty$, so that $|0|_p = 0$. For $x \in \mathbb{Q}$, we also define $\text{ord}_p(x) = v_p(x)$.

This extends to \mathbb{Q}_p . If $x \in \mathbb{Q}_p$ and we write x in the standard p -adic representation $x = \sum_{i=N}^{\infty} a_i p^i$ ($N \in \mathbb{Z}$, the a_i 's are integers with $0 \leq a_i < p$ and $a_N \neq 0$), then we define $\text{ord}_p(x) = N$ and $|x|_p = p^{-N}$. Clearly, in the special case $x \in \mathbb{Q}$, these definition agree with those given above.

More generally, if $x \in E_p$, where E_p is a finite extension of \mathbb{Q}_p , of degree, say d , and $t^d + b_{d-1}t^{d-1} \dots + b_1t + b_0$ is the characteristic polynomial of x with respect to the extension E_p/\mathbb{Q}_p , ($b_i \in \mathbb{Q}_p$ for $i = 0, \dots, d-1$), then

$$\text{ord}_p(x) = \frac{1}{d} \text{ord}_p(b_0) = \frac{1}{d} \text{ord}_p(N_{E_p/\mathbb{Q}_p}(x)) \quad \text{and} \quad |x|_p = p^{-\text{ord}_p(x)}. \quad (\text{A.1})$$

These definitions are independent from E_p ; in particular, they coincide with the definitions of $\text{ord}_p(x)$ and $|x|_p$ given at the beginning with $x \in \mathbb{Q}_p$.

Now we adopt a different point of view. Let

$$E = \mathbb{Q}(\xi), \quad \text{where } g(\xi) = 0 \text{ and } g(t) \in \mathbb{Q}[t] \text{ is monic and irreducible.}$$

We denote by \mathcal{O}_E the maximal order of E . Let

$$p \mathcal{O}_E = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m} \quad (\text{A.2})$$

be the factorization of the principal ideal $p \mathcal{O}_E$ into prime ideals of E , where the \mathfrak{p}_i 's above are distinct and ramification index $e_{E/\mathbb{Q}}(\mathfrak{p}_i) = e_i > 0$ for every $i = 1, \dots, m$; we also denote by f_i the residual degree $f_{E/\mathbb{Q}}(\mathfrak{p}_i)$.

For every $x \in E$ and every \mathfrak{p}_i we denote by $v_{\mathfrak{p}_i}(x)$ the exponent of \mathfrak{p}_i in the prime ideal factorization of $x \mathcal{O}_E$; in particular, $v_{\mathfrak{p}_i}(p) = e_i$. If \mathfrak{p}_i is a principal ideal, say $\mathfrak{p}_i = \pi \mathcal{O}_E$, then we write $v_{\pi}(x)$.

The polynomial $g(t)$ factorizes into m distinct irreducible polynomials of $\mathbb{Q}_p[t]$:

$$g(t) = g_1(t) \cdots g_m(t). \quad (\text{A.3})$$

Let

$$E_{\mathfrak{p}_i} = \mathbb{Q}_p(\xi_{\mathfrak{p}_i}), \text{ where } \xi_{\mathfrak{p}_i} \text{ is defined by } g_i(\xi_{\mathfrak{p}_i}) = 0;$$

actually, $E_{\mathfrak{p}_i}$ is the completion of $(E, |\cdot|_{\mathfrak{p}_i})$, where $|\cdot|_{\mathfrak{p}_i}$ is the absolute value of E corresponding to the additive valuation $v_{\mathfrak{p}_i}(\cdot)$. There is a natural embedding $E \xrightarrow{\phi_i} E_{\mathfrak{p}_i}$ mapping ξ to $\xi_{\mathfrak{p}_i}$, which allows us to view E as a subfield of $E_{\mathfrak{p}_i}$. The typical element $x(\xi) \in E$ (where $x[t] \in \mathbb{Q}[t]$) can be viewed as an element of $E_{\mathfrak{p}_i}$ if we identify $x(\xi)$ with $\phi_i(x(\xi))$. Formally, this means that we view $x(\xi)$ as the element $x(\xi_{\mathfrak{p}_i}) = x(t) + g_i(t)\mathbb{Q}_p[t] \in \mathbb{Q}_p[t]/g_i(t)\mathbb{Q}_p[t]$. Then, according to (A.1),

$$\text{ord}_p(x(\xi)) = \frac{1}{[\mathbb{Q}_p(\xi_{\mathfrak{p}_i}) : \mathbb{Q}_p]} \text{ord}_p(N_{\mathbb{Q}_p(\xi_{\mathfrak{p}_i})/\mathbb{Q}_p}(x(\xi_{\mathfrak{p}_i}))) \quad \text{and} \quad |x(\xi)|_p = p^{-\text{ord}_p(x(\xi))}. \quad (\text{A.4})$$

The above discussion makes clear that the value of $\text{ord}_p(x(\xi))$ depends on \mathfrak{p}_i . Consequently, if $i \neq j$ and $x(\xi) \in E$, then, the value of $\text{ord}_p(x(\xi))$ may vary, depending on whether we view E as a subfield of $E_{\mathfrak{p}_i}$ or of $E_{\mathfrak{p}_j}$.

The enumeration of the \mathfrak{p}_i 's in (A.2) and the g_i 's in (A.3) can be done in such a way that

$$\deg g_i = [E_{\mathfrak{p}_i} : \mathbb{Q}_p] = e_i f_i \quad \text{and} \quad v_{\mathfrak{p}_i}(x(\xi)) = e_i \cdot \text{ord}_p(x(\xi)). \quad (\text{A.5})$$

The second relation above implies that, for the typical element $x(\xi) \in E$ (where $x[t] \in \mathbb{Q}[t]$), the following is true: $x(\xi)$ is divisible by \mathfrak{p}_i iff $\text{ord}_p(x(\xi_{\mathfrak{p}_i})) > 0$ which, in turn, is equivalent to the statement that the constant term of the characteristic polynomial of $x(\xi_{\mathfrak{p}_i})$ over \mathbb{Q}_p has positive ord_p . Having established this enumeration, we have for every $x(\xi) \in E$:

$$v_{\mathfrak{p}_i}(x(\xi)) = \frac{e_{E/\mathbb{Q}}(\mathfrak{p}_i)}{[E_{\mathfrak{p}_i} : \mathbb{Q}_p]} \text{ord}_p(N_{E_{\mathfrak{p}_i}/\mathbb{Q}_p}(x(\xi_{\mathfrak{p}_i}))) = e_{E/\mathbb{Q}}(\mathfrak{p}_i) \cdot \text{ord}_p(x(\xi)). \quad (\text{A.6})$$

In practice, the above mentioned correspondence $\mathfrak{p}_i \leftrightarrow g_i$ is carried out by a MAGMA routine which we wrote based on the following: For $j = 1, \dots, m$, consider the “two-element representation” of \mathfrak{p}_j , namely, $\mathfrak{p}_j = p\mathcal{O}_E + h_j(\xi)\mathcal{O}_E$, where $h_j(t) \in \mathbb{Q}[t]$, and fix any $i \in \{1, \dots, m\}$. For $j = 1, \dots, m$, compute $\phi_i(h_j(\xi)) = h_j(\xi_{\mathfrak{p}_i})$ and the characteristic polynomial $\chi_j(t)$ of $h_j(\xi_{\mathfrak{p}_i})$ with respect to the extension $E_{\mathfrak{p}_i}/\mathbb{Q}_p$. For exactly one index j the ord_p of the constant term of $\chi_j(t)$ is positive. The polynomial $g_j(t)$, for this specific j , corresponds to the ideal \mathfrak{p}_i . This we do for any $i = 1, \dots, m$ and we establish the one-to-one correspondence $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\} \leftrightarrow \{g_1, \dots, g_m\}$. By permuting the indices of g_1, \dots, g_m , if necessary, we establish the one-to-one correspondence $\mathfrak{p}_i \leftrightarrow g_i$ which satisfies (A.5) and (A.6).

Appendix B

Working in K .

In order to apply the method of [34] we need to work in an extension K of F in which $g(t)$ has at least three distinct roots. Thus, in general, we do not need the whole splitting field of $g(t)$ over F . In our case, however, the Galois group of $g(t)$ is of order 20, which implies that K is the splitting field of $g(t)$ over F . Using MAGMA we find out that $K = \mathbb{Q}(\omega)$, where ω is a root of the polynomial

$$\begin{aligned} G(t) = & t^{20} + 780t^{19} + 248030t^{18} + 39929580t^{17} + 3046440525t^{16} + 18210793968t^{15} - 13729990391320t^{14} \\ & + 752551541981520t^{13} + 8605950990819730t^{12} + 1708764818389209000t^{11} \\ & + 23308084571944423284t^{10} - 1404817549102176551640t^9 - 35442768652652017430190t^8 \\ & + 375805034836819117590960t^7 + 16191084883780784798260200t^6 \\ & + 30210122048192693893581552t^5 - 2113554835538935196795743635t^4 \\ & - 12364486598313473303834175060t^3 + 25061666765667764525027943390t^2 \\ & + 278757784774895111708136641100t + 427756623168133431059207412321. \end{aligned}$$

Of course, K is a Galois extension. By \mathcal{O}_K we denote the maximal order of K and by \mathcal{O}_F the maximal order of F . The roots $\theta^{(i)}$ ($i = 1, \dots, 5$) of $g(t)$ are polynomial expressions of ω with rational coefficients. Thus $\theta^{(i)} = \theta_i(\omega) \in \mathbb{Q}[\omega]$, where $\theta_i(t) \in \mathbb{Q}[t]$ ($i = 1, \dots, 5$). Each $\theta^{(i)}$ corresponds to an embedding $\psi_i : F \hookrightarrow K$ characterized by $\psi_i(\theta) = \theta_i(\omega)$. Then we can view F as subfield of K in five ways, by identifying F with $\psi_i(F)$. For our computations we can arbitrarily choose the embedding ψ_i , but, once we choose it, we must keep it fixed. MAGMA computes the embeddings ψ_i ; rather arbitrarily, when it considers F as a subfield of K , our MAGMA session implicitly uses the embedding ψ_5 ; i.e. as a subfield of K , F is identified with $\psi_5(F)$. For simplicity, we set $\psi = \psi_5$ and avoid the use of superscript/subscript indicating the conjugation. Then, extending a prime ideal \mathfrak{p} of \mathcal{O}_F to the ideal $\mathfrak{p}\mathcal{O}_K$ means, the following: Let $\mathfrak{p} = p\mathcal{O}_F + h(\theta)\mathcal{O}_F$, where $h(\theta) \in \mathbb{Q}[\theta]$. Then $\mathfrak{p}\mathcal{O}_K = p\mathcal{O}_K + h(\psi(\theta))\mathcal{O}_K$. MAGMA computes $\psi(\theta) = \psi_5(\theta) = \theta^{(5)} = \theta_5(\omega)$:

$$\begin{aligned} \theta_5(\omega) = & (109949833761153867182006233162830218735318443\omega^{19} \\ & + 85209181362831884900908863253209941719229099772\omega^{18} \\ & + 26843948030443532446996152295591813313759057989280\omega^{17} \\ & + 4256032326064110742980882546282193567817773030947507\omega^{16} \\ & + 313740861737738480045017056015360916332191517809963652\omega^{15} \\ & + 448056667342462133472342261671551993343950346279488008\omega^{14} \\ & - 1510732666058952542367528947242709391406430475149001889048\omega^{13} \\ & - 75203582469112061453856831841769761124516215292521423604700\omega^{12} \\ & + 1315665925830026167841281970821852638550108379556176872190482\omega^{11} \\ & + 181102009414236143682774128780867507407780245434648419322814672\omega^{10} \\ & + 1669796300877028395254905627635289010395439563221417145651022616\omega^9 \\ & - 162206235802302893664200291374667388423732969287959076428108697406\omega^8 \end{aligned}$$

(continued on next page)

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$$\begin{aligned}
& - 3096949958783354342722100732857399558416026343530279487756804602460 \omega^7 \\
& + 56131367252940396214847119246725578907696249119733671305047923800344 \omega^6 \\
& + 1504180072707173996628600619493969793943259186258161149867325499759064 \omega^5 \\
& - 3870840446117907139295078483933668999054374262639172467797695558733756 \omega^4 \\
& - 213744087631757677190145693131224085047423871295167330938194597435156069 \omega^3 \\
& - 352243194516613905751693267069064334504141157799975445500242914720886380 \omega^2 \\
& + 4494640381465135578890041723352987174508368793027744613522589459354871816 \omega \\
& + 10018536620467342924854600525808560481514235129562992768169198074640026531)/ \\
& \quad 2375073599455257025973891103597991836268467069206252327298662400000000.
\end{aligned}$$

For the ideal $\mathfrak{p} = \pi_{113} \mathcal{O}_F$, obviously, we can take $h(\theta) = \pi_{113}$ (see page 11). Then

$$\begin{aligned}
\psi(h(\theta)) = \psi(\pi_{113}) = & (4378482585825381431566239057028627160970028037 \omega^{19} \\
& + 3396229499207087892836807551803460134822567613017 \omega^{18} \\
& + 1071214373586764983689778732938427724586732483076327 \omega^{17} \\
& + 170145388685753978342778727714777154218418254512908043 \omega^{16} \\
& + 12588645126248857287103560864595388726070407650893086772 \omega^{15} \\
& + 23291648519661572328486403102949754117988678982445451332 \omega^{14} \\
& - 60338282775505500617169124951383242755661107414046130424148 \omega^{13} \\
& - 3031547893595987199227070326893238313728050147362743168899908 \omega^{12} \\
& + 51530222400182570158994336776484627607851706804827046320630198 \omega^{11} \\
& + 7282924550357450053556161972634939332060481531424832236222347982 \omega^{10} \\
& + 69407725895185343408814814665447944190629387800816493362801270434 \omega^9 \\
& - 6522081891215060359860869744209143119462208244901441175704137508646 \omega^8 \\
& - 126498537591831789061901120743476135204565837943425968840309704817756 \omega^7 \\
& + 2266682290068533497159436879932180463046527604540137185016229550764276 \omega^6 \\
& + 61296290267956300004998582874965020844943081524921068036926148986534364 \omega^5 \\
& - 165499773355462341124868225890607890561889109853343804837744925708526164 \omega^4 \\
& - 8698248530644221957092224575657663153179204605475867759713929659343969939 \omega^3 \\
& - 11520188167294592917194821184972443593645584955137753402305064016274508639 \omega^2 \\
& + 173158801650947255768584220314522267452080115712427401970346621418344586479 \omega \\
& + 378768220799897336478183837425696328863796308850264844412883432915449692851)/ \\
& \quad 152004710365136449662329030630271477521181892429200148947114393600000000.
\end{aligned}$$

Our MAGMA routine, mentioned below the equation (A.6), returns the factorization $11\mathcal{O}_K = \prod_{i=1}^{10} \mathfrak{P}_i^2$, where the residual degree $f_{K/\mathbb{Q}}(\mathfrak{P}_i) = 1$ for every $i = 1, \dots, 10$. Moreover, for every $i = 1, \dots, 10$, the routine computes:

- An element $h_i \in \mathcal{O}_K$, such that $\mathfrak{P}_i = 11\mathcal{O}_K + h_i\mathcal{O}_K$.
- The factorization of $G(t) = \prod_{i=1}^{10} G_i(t)$ into irreducible polynomials over \mathbb{Q}_p . For $i = 1, \dots, 10$, the irreducible polynomial $G_i(t) \in \mathbb{Q}_p[t]$ corresponds to \mathfrak{P}_i in the sense explained in Appendix A. As expected, $\deg G_i(t) = 2$ for every $i = 1, \dots, 10$.

In table 6 we give the data mentioned in the above two “bullets”. The element h_i is identified with a 20-tuple: $h_i = (c_{i1}, \dots, c_{i20})$ means that

$$h_i = \sum_{j=1}^{20} c_{ij} \beta_j,$$

where $\beta_1, \dots, \beta_{20}$ is an integral basis of K/\mathbb{Q} , explicitly calculated by MAGMA. For the polynomial $G_i(t)$ we write $G_i(t) = (\gamma_{i1}, \gamma_{i0})$, by which we mean that $G_i(t) = t^2 + \gamma_{i1}t + \gamma_{i0}$. In the columns of the γ_{ij} ’s we write their 11-adic approximations, (rational integers) with precision $O(11^{10})$.

Table 6: $11\mathcal{O}_K = \prod_{i=1}^{10} \mathfrak{P}_i^2$

	$\mathfrak{P}_i = 11\mathcal{O}_K + h_i\mathcal{O}_K$	$K_{\mathfrak{P}_i} = \mathbb{Q}_{11}[t]/\langle t^2 + \gamma_{i1}t + \gamma_{i0} \rangle$	
i	h_i	γ_{i1}	γ_{i0}
1	(5, 8, 10, 5, 3, 3, 7, 0, 5, 5, 0, 3, 2, 10, 0, 2, 10, 2, 1, 7)	11244468595	6668815422
2	(6, 2, 0, 0, 7, 10, 4, 0, 6, 6, 4, 5, 2, 10, 0, 2, 10, 2, 1, 7)	−169320583	10491152974
3	(6, 9, 8, 0, 6, 6, 1, 5, 2, 0, 6, 5, 2, 10, 0, 2, 10, 2, 1, 7)	−9236124994	5583083423
4	(1, 5, 8, 3, 7, 5, 10, 6, 9, 0, 9, 3, 2, 10, 0, 2, 10, 2, 1, 7)	−6126278749	7582171800
5	(2, 1, 10, 4, 6, 7, 8, 6, 2, 9, 1, 3, 7, 6, 5, 10, 7, 2, 10, 7)	10744341441	−10666285673
6	(8, 4, 10, 3, 5, 4, 3, 4, 4, 1, 5, 3, 4, 0, 8, 2, 1, 0, 8, 7)	−3779293982	290904043
7	(10, 8, 2, 1, 0, 9, 10, 4, 8, 4, 4, 8, 10, 2, 1, 7, 9, 4, 8, 7)	−669447737	7802303026
8	(10, 1, 10, 2, 6, 10, 6, 7, 2, 1, 1, 0, 5, 1, 6, 10, 5, 3, 8, 7)	−12083236915	−10106323842
9	(8, 7, 9, 1, 9, 10, 10, 3, 3, 0, 8, 10, 10, 4, 6, 6, 6, 8, 8, 7)	10744341441	9625552201
10	(6, 0, 10, 3, 6, 2, 7, 9, 3, 7, 6, 1, 8, 10, 0, 1, 9, 1, 2, 9)	−669447737	−12489534848

Now, for $\mathfrak{p} = \pi_{113}\mathcal{O}_F$ we have to know the factorization of $\mathfrak{p}\mathcal{O}_K$; of course, the prime ideals of \mathcal{O}_K in this factorization belong to $\{\mathfrak{P}_1, \dots, \mathfrak{P}_{10}\}$. For this purpose it suffices to compute $v_{\mathfrak{P}_i}(\psi(\pi_{113}))$ for $i = 1, \dots, 10$. This we do easily using MAGMA. We find out that $v_{\mathfrak{P}_i}(\psi(\pi_{113})) = 0$ for $i = 1, \dots, 8$ and $v_{\mathfrak{P}_i}(\psi(\pi_{113})) = 2$ for $i = 9, 10$; hence $\pi_{113}\mathcal{O}_F = \mathfrak{P}_9^2\mathfrak{P}_{10}^2$. According to the above we put $\mathfrak{P} = \mathfrak{P}_9$, so that

$$e_{K/\mathbb{Q}}(\mathfrak{P}) = 2, \quad f_{K/\mathbb{Q}}(\mathfrak{P}) = 1 \tag{B.1}$$

$$G_{\mathfrak{P}}(t) = G_9(t) = t^2 + \gamma_{91}t + \gamma_{90}.$$

Working p -adically in K means working in $K_{\mathfrak{P}} = \mathbb{Q}_{11}(\omega_{\mathfrak{P}}) \cong \mathbb{Q}_{11}[t]/\langle G_{\mathfrak{P}}(t) \rangle$, where each root $\theta^{(i)}$ ($i = 1, \dots, 5$) is identified with $\theta_i(\omega_{\mathfrak{P}})$.

We have $g_3(\theta_5(\omega_{\mathfrak{P}})) = 0$. Indeed, we have the following commutative diagram of monomorphisms:

$$\begin{array}{ccc} F & \xrightarrow{\psi=\psi_5} & K \\ \downarrow \phi & & \downarrow \Phi \\ F_{\mathfrak{p}} & \xrightarrow{\Psi} & K_{\mathfrak{P}} \end{array}$$

Here $F_{\mathfrak{p}} = \mathbb{Q}_p(\theta_{\mathfrak{p}})$, where $\theta_{\mathfrak{p}} = 7 + 2 \cdot 11 + 2 \cdot 11^2 + 10 \cdot 11^3 + 7 \cdot 11^4 + \dots \in \mathbb{Q}_{11}$, is the root of $g_3(t)$.¹⁸ The natural embeddings ϕ and Φ are in accordance with the general discussion a few lines below the relation (A.3). Thus, $\phi(\theta) = \theta_{\mathfrak{p}}$, $\Phi(\omega) = \omega_{\mathfrak{P}}$ and, consequently,

$$\Psi(\theta_{\mathfrak{p}}) = \Phi \circ \psi_5 \circ \phi^{-1}(\theta_{\mathfrak{p}}) = \Phi \circ \psi_5(\theta) = \Phi(\theta_5(\omega)) = \theta_5(\Phi(\omega)) = \theta_5(\omega_{\mathfrak{P}}).$$

Therefore, $g_3(\theta_5(\omega_{\mathfrak{P}})) = g_3(\Psi(\theta_{\mathfrak{p}})) = \Psi(g_3(\theta_{\mathfrak{p}})) = \Psi(0) = 0$. Further, using MAGMA we see that the (11-adic) roots of $g_1(t)$ are $\theta_i(\omega_{\mathfrak{P}})$ with $i = 2, 4$, and the roots of $g_2(t)$ are $\theta_i(\omega_{\mathfrak{P}})$ with $i = 1, 3$.

In Subsection 3.2.5, where we view (3.17) as a relation in $K_{\mathfrak{P}}$ (which simply means that we apply Φ to (3.17)) we will choose $i_0 = 5$, $j = 1$ and $k = 3$, following the instructions at bottom of p. 235 of [34].

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¹⁸We remind that $g_1(t), g_2(t), g_3(t)$ are defined at the beginning of Subsection 3.2.3.

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